A Method for 4x4 Digit Mental Multiplications

This paper contains a method for mentally calculating 4-digit by 4-digit multiplications without writing down any intermediate values. For writing down digits as the calculation proceeds, cross-multiplication may be faster, although it produces digits in a right-to-left manner. A more impressive feat is to announce the entire answer in a left-to-right manner without writing or saying any intermediate values. Assuming that 2x2 digit multiplications can be done mentally (even if slowly), the procedure presented here provides a convenient method that minimizes the number of intermediate values to remember.

First, we will need the number grouping concept from my book, ***Dead Reckoning: Calculating Without Instruments***. Since we would like to work with individual two-digit multiplications, it is convenient to treat hundreds groupings as separate blocks. So the notation "**|n**" represents a two-digit number string. If more than two digits exist in **n**, they are merged (or added) to the digits to the left of the "**|**" sign. For example, 3|129 = 4|29 = 429. We simply want to work with hundreds groups in a number, and these can carry or borrow as needed from neighboring groups. So if we end up with 25|-125, we can borrow 2 from the leftmost group to make the rightmost one positive, so we convert this to 23|75 or 2375 when we drop the vertical bar at the end. To summarize, if we get 34|145|16|-248 from doing separate two-digit math operations, it would become 35|45|13|52, or 35451252 to get a final answer that merges all the parts.

If you wish, you can skip the derivation that is next and go directly to ***The Steps*** further down to see how it works.

The Derivation

Assume that A, B, C, and D represent two-digit numbers, so a four-digit number can be represented as A|B or C|D. AC means A•C, and CD means C•D.

For a 4x4 multiplication, the normal expansion of the partial products using two-digit multiplications is

$$A|B \cdot C|D = AC \cdot 10^4 + (AD + BC) \cdot 10^2 + BD$$

which requires four two-digit multiplications, plus shifting and adding.

There is a way given by Knuth[1] to do this with only three two-digit multiplications. Using my notation,

$$A|B \cdot C|D = AC \cdot 10^4 + [AC + BD - (A-B)(C-D)] \cdot 10^2 + BD$$

or:

$$A|B \cdot C|D = AC \mid [AC + BD - (A-B)(C-D)] \mid BD$$

This requires keeping some numbers in memory for a time. However, the memory requirements are simplified if we note that if **p** and **q** are single digits in a two-digit number,

$$101 \cdot (10p + q) = p \cdot 10^4 + (p+q) \cdot 10^2 + q$$

so:

$$101 \cdot A|B = AC \mid AC+BD \mid BD$$

and we have:

$$A|B \cdot C|D = 101 \cdot AC|BD \; - \; 100 \cdot (A-B)(C-D)$$

These pieces get merged into the two-digit groupings as we proceed with the calculation.

The Steps

So here are the steps, using an example of 6143 • 2839 or 61|43 • 28|39 in place of A|B • C|D, and assuming that these numbers are in view so we don't have to memorize them, as below:

6143 • 2839

1. Find AC = 61•28 = 1708. Find BD = 43•39 = 1677.

2. Merge them as AC|BD: 1708 | 1677 = 17 | 24 | 77

At this point, all we have to remember is this set of three numbers (17 | 24 | 77). There is no need to remember anything from step 1.

3. Now let's take the sum of the last two numbers, 24+77, and subtract (A-B)•(C-D). Glancing, we see that A>B but C<D, so we can just take the positive differences and change the subtraction to an addition:

24 + 77 + (61-43)•(39-28) = 101 + 11•18 = 299

We need to remember this number, 299, too.

4. Now the answer is:

17 | (17 + 24) | 299 | 77

where the first number is the leftmost number of step 2, the second number is the sum of the leftmost and middle number of step 2, the third number is the one we calculated in step 3, and the fourth number is the rightmost number in step 2.

We don't need to have all at once in our head--we create and merge it, and look ahead to correct for carries or borrows, as we write it down from left to right, the first time we write anything down:

17439977

which is the correct answer.

Another Example

Let's do this for an example where everything is the most complicated that we can get for this method, i.e., where there are carries and borrows and the third term has a large subtraction.

Find 6137 • 8548

1. Find AC = 61•85 = 5185. Find BD = 37•48 = 1776.

2. Merge them as AC|BD: 5185 | 1776 = 51 | 102 | 76 = 52 | 2 | 76

At this point, all we have to remember is this set of three numbers (52 | 2 | 76). There is no need to remember anything from step 1.

3. Now let's take the sum of the last two numbers, 2+76, and subtract (A-B)•(C-D). Glancing, we see that A>B and C>D, so we take the positive differences but keep it a subtraction:

78 - (61-37)•(85-48) = 78 - 24•37 = -810

We need to remember this number, -810, too.

4. Now the answer is:

52 | (52 + 2) | -810 | 76

and we merge it as we write it down (note that we have to take 9 from the second term to add 900 to the third term to make it positive):

52459076

which is again the correct answer.

Conclusion

The method given in this paper for mentally calculating 4x4 digit multiplications offers the following advantages:

1. There are only three two-digit multiplications instead of the four from standard partial products.
2. The shifting of the partial products is mechanical and easy.
3. There are very few intermediate numbers to remember.
4. It is ideal if there is little difference between the first two digits and the second two digits of one of the numbers.
5. The answer can be read out from left to right, with no intermediate results, which is more impressive than the right-to-left output of cross-multiplication.

[1]Donald E. Knuth, ***The Art of Computer Programming Vol. II: Seminumerical Algorithms***, Addison-Wesley, Reading, 1969.