A Square Root Algorithm Example

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My book, ***Dead Reckoning: Calculating Without Instruments***, provides a general algorithm and examples for mentally calculating the decimal square root of a number. The following is an additional, more explanatory example of calculating the square root of 416291 using this method.

First, I need to describe the number grouping concept from the book. Since it is feasible to work with two-digit multiplications, it is convenient to treat hundreds groups as separate groups. So the notation "|n" represents a two-digit number string. If more than two digits exist in n, they are merged (added) to the digits to the left of the "|" sign. For example, 3|129 = 4|29 = 429. We simply want to work with hundreds groups in a number, and these can carry or borrow as needed from neighboring groups. So if we end up with 25|-125, we can borrow 2 from the leftmost group to make the rightmost one positive, so we convert this to 23|75 and to 2375 when we drop the vertical bar at the end. To summarize, if we end up with 34|145|16|-48, it would become 35|45|15|52, or 35451552 as the final answer that merges all the sets. This merging is fast and can be done when the answer is being written down.

First, for simplicity move the decimal point two places at a time so the value to the left of the decimal point will have a two-digit root—we will move the decimal point to the right one place at a time in the final square root to correct for this initial shift. Then take this value and find the nearest square root and remainder:

N = 416291 => 4162.91
$4162 - 64^2 = 66$

So the current estimate is 64 with a remainder of 66.

Take 100 times the remainder 66 and divide by 2, then add ½ the next two digits of N, and then divide by 64 (in the first step here, you can just set 6600/2 + 91/2 to 6691/2):

$$\frac{6691/2}{64} = 52 \text{ R } 17.5$$ where R indicates the remainder

Current value: 64 | 52

Take 100 times the remainder 17.5, then add ½ the next two digits of N (there are none in this example), then subtract $52^2/2$, and then divide by 64:

$$\frac{1750 - 52^2/2}{64} = 6 \text{ R}14$$

Current value: 64 | 52 | 06

Take 100 times the remainder 14, then add ½ the next two digits of N (there are none in this example), then subtract 52*6, and then divide by 64:

$$\frac{1400 - 52(6)}{64} = 17 \text{ R0}$$

Current value: 64 | 52 | 06 | 17

Here we arrive at an 8-digit accuracy with a very minimum of mental calculation. Merging these sets and fixing the decimal point we get $41629^{1/2} = 645.20617\ldots$

Let's continue so that the whole pattern of operations becomes clearer. The algorithm after finding the initial two-digit estimate and remainder (64 and 66 here) is:

> 1. Take 100 times the remainder and divide by 2, then add ½ the next two digits of N if they exist. In our example we this was (6600 + 19)/2 = 6619/2. We could also have chosen an initial estimate of 65 rather than 64, and then our initial remainder would have been -63, and in this case we would have had (-6300 + 19)/2 = -6281/2. Divide by the initial estimate (64) to get the next two-digit number (a group) and remainder.

Then repeat steps 2 to 4 until you decide to quit or can't remember all the digits:

> 2. Take 100 times the remainder from the previous step and add ½ the next two digits of N if they exist. Consider the groups of two-digit numbers following the original estimate (or 64 here). Starting from the outside two groups of these, multiply them pairwise as you work inward to the middle. Subtract each of these products. (There will be only one group after the first estimate the first time, so skip this and go to step 3 below.)
>
> 3. If there is a number left in the middle after the group pairs are multiplied working in from the outside, subtract the square of that number divided by 2.
>
> 4. Divide by the original two-digit estimate to get a new number for the next group and a new remainder. We can adjust the quotient to get a negative remainder if we think it will make the numerator in the next step smaller and the magnitude of the next number group less than or equal to 50. We already have one group of 52, but that is relatively close to 50.

When you quit, merge the sets back into a single number and place the decimal point in the correct position.

So, watch for this pattern as we continue where we left off:

Current value: 64 | 52 | 06 | 17 and our previous remainder was 0

$$\frac{00 - 52(17) - 6^2/2}{64} = -(14 \text{ R6}) = -14 \text{ R-6}$$

Current estimate: 64 | 52 | 06 | 17 | -14

$$\frac{-600 - 52(-14) - 6(17)}{64} = 0 \text{ R26}$$

Current value: 64 | 52 | 06 | 17 | -14 | 00

$$\frac{2600 - 52(0) - 6(-14) - 17^2/2}{64} = 39 \text{ R43.5 or with less error, } 40 \text{ R-20.5}$$

Current value: 64 | 52 | 06 | 17 | -14 | 00 | 40

$$\frac{-2050 - 52(40) - 6(0) - 17(-14)}{64} = -(60 \text{ R } 52) = -(61 \text{ R } -12) = -61 \text{ R12}$$

Current value: 64 | 52 | 06 | 17 | -14 | 00 | 40 | -61

Let’s stop here. Merging the sets gives us:

$416291^{1/2} = 645.2061686003939\ldots$

which is accurate to this last digit we calculated it to. If the last digit were in error by one, the next iteration would fix it.

This algorithm is more complicated to explain than it is to remember it once you understand it. Don’t merge the sets until the very end or this method won’t work. If you think this is a lot of effort, you might try getting this many digits using the old pencil-and-paper method...