The Square Root of 121432

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My book, ***Dead Reckoning: Calculating Without Instruments***, provides a general algorithm and examples for mentally calculating the decimal square root of a number. The following is an additional, more explanatory example of calculating the square root of 121432 using this method. Note that in order to be very clear, this description looks a lot more complicated than it actually is when doing the calculation mentally; it becomes much easier when the steps become natural to you.

First, I need to describe the number grouping concept from the book. Since it is feasible to work with two-digit multiplications, it is convenient to treat hundreds groups as separate groups, or sets. So the notation "|n" represents a two-digit number string. If more than two digits exist in n, they are merged (added) to the digits to the left of the "|" sign. For example, 3|129 = 4|29 = 429. We simply want to work with hundreds groups in a number, and these can carry or borrow as needed from neighboring groups. So if we end up with 25|-125, we can borrow 2 from the leftmost group to make the rightmost one positive, so we convert this to 23|75 and to 2375 when we drop the vertical bar at the end. To summarize, if we end up with 34|145|16|-48, it would become 35|45|15|52, or 35451552 as the final answer that merges all the sets. This merging is fast and can be done when the answer is being written down.

First, for simplicity move the decimal point two places at a time so the value to the left of the decimal point will have a two-digit root—we will move the decimal point to the right one place at a time in the final square root to correct for this initial shift. In this case, even though 121.432 has a two-digit square root, we can only move the decimal point a multiple of two places, so we have to make it 1214.32, which also has a two-digit root.

Then we take the value to the left of the decimal point and find the nearest square root and remainder:

N = 121432 => 1214.32
$1214 - 35^2 = -11$

So the current estimate is 35 with a remainder of -11 = 35 R -11, where R indicates the remainder.

Take 100 times the remainder -11 and divide by 2, then add ½ the next two digits of N, and then divide by 35. We want the nearest two-digit value, so R can be negative or positive.

$$\frac{-1100/2 + 32/2}{35} = \frac{-534}{35} = -(15 \text{ R } 9) = -15 \text{ R } -9$$

Current value: 35 | -15

Take 100 times the remainder -9, then add ½ the next two digits of N (there are none in this example), then subtract $(-15)^2/2$, and then divide by 35. Again, we find a quotient that gives the smallest absolute value of R.

$$\frac{-900 - (-15)^2/2}{35} = \frac{-1012.5}{35} = -(29 \text{ R } -2.5) = -29 \text{ R } 2.5$$

Current value: 35 | -15 | -29

Take 100 times the remainder 2.5, then add ½ the next two digits of N (there are none in this example), then subtract (-15)x(-29), and then divide by 35:

$$\frac{250 - (-15)(-29)}{35} = \frac{-185}{35} = -(5 \text{ R } 10) = -5 \text{ R } -10$$

Current value: 35 | -15 | -29 | -05

Here we arrive at an 8-digit accuracy with a very minimum of mental calculation (we've mostly been talking). Merging these sets and fixing the decimal point we get $121432^{1/2} = 348.47095\ldots$

Merging number sets when there are neighboring negative numbers is not hard once you see how it is done. Remember that we can subtract 1 from the neighbor to the left to "borrow" an amount of 100 to make a negative number group positive. So the steps that we mentally go through as we merge the number sets are:

35 | -15 | -29 | -05 $\rightarrow$ 3485 | -29 | -05 $\rightarrow$ 348471 | -05 $\rightarrow$ 34847095

After a bit of practice, merging number groups of various mixtures of positive and negative number groups becomes easy as we recite the final result from left to right.

Let's continue so that the whole pattern of operations becomes clearer. The algorithm after finding the initial two-digit estimate and remainder (35 and -11 here) is:

> 1. Take 100 times the remainder and divide by 2, then add ½ the next two digits of N if they exist. In our example we this was (-1100 + 32)/2 = -1181/2. Divide by the initial estimate (35) to get the next two-digit number set and remainder.

Then repeat steps 2 to 4 until you decide to quit or can't remember all the digits:

> 2. Take 100 times the remainder from the previous step and add ½ the next two digits of N if they exist. Consider the sets of two-digit numbers following the original estimate (or 35 here). Starting from the outside two sets of these, multiply them pairwise as you work inward to the middle. Subtract each of these products. (There will be only one set after the first estimate the first time, so skip this and go to step 3 below.)
>
> 3. If there is a number left in the middle after the set pairs are multiplied working in from the outside, subtract the square of that number divided by 2.
>
> 4. Divide by the original two-digit estimate to get a new number for the next set and a new remainder. We can adjust the quotient to get a negative remainder if we think it will make the numerator in the next step smaller and the magnitude of the next set less than or equal to 50.

When you quit, merge the sets back into a single number and place the decimal point in the correct position.

So, watch for this pattern as we continue where we left off:

Current value: 35 | -15 | -29 | -05 and our previous remainder was -10

$$\frac{-1000 - (-15)(-5) - (-29)^2/2}{35} = \frac{-1495.5}{35} = -(43 \text{ R } -9.5) = -43 \text{ R } 9.5$$

Current value: 35 | -15 | -29 | -05 | -43

$$\frac{950 - (-15)(-43) - (-29)(-5)}{35} = \frac{160}{35} = 5 \text{ R } -15$$

Current value: 35 | -15 | -29 | -05 | -43 | 05

$$\frac{-1500 - (-15)(5) - (-29)(-43) - (-5)^2/2}{35} = \frac{-2684.5}{35} = -(76 \text{ R } 24.5)$$

But this quotient of 76 is significantly larger than 50, and we want to keep the number groups near or below 50 so that they don't have as much ripple effect when calculating later number sets. So let's back up and change the previous result from 5 R -15 to 4 R 20, which will increase the numerator in this step by (3500-15) to get 800.5 and we now get a result of 23 R -4.5 for this step.

Current value: 35 | -15 | -29 | -05 | -43 | 04 | 23

$$\frac{-450 - (-15)(23) - (-29)(4) - (-5)(-43)}{35} = \frac{-204}{35} = -(6 \text{ R } 6) = -6 \text{ R } -6$$

Current value: 35 | -15 | -29 | -05 | -43 | 04 | 23 | -06

Let's stop here. Merging the sets and placing the final decimal point gives us:

$121432^{1/2} = 348.4709457042294\ldots$

which is a 16-digit square root accurate to the last digit we calculated it to. If the last digit were in error by one, the next iteration would fix it. Remember, don't merge the number sets until the very end or this method won't work.

Again, this algorithm is more complicated to explain than it is to remember it once you understand it. And, of course, there's no reason to calculate the square root to as many decimal places as we have done here—this is an extreme example to show how to handle a variety of situations. On the other hand, I have found that solutions to lower accuracy only seem easy after spending some time attempting them to higher accuracy.