Notes and Errata for the Book

Dead Reckoning: Calculating Without Instruments

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Version Release: September 5, 2006

Types:

- **Note**: A clarification or elaboration of the text.
- **Typo**: A simple mistake that does not affect the method presented.
- **Error**: An error that may affect a method or the reader's interpretation of it.

Note: An asterisk (*) next to the page number indicates a new or updated item since the last released version of this document (June 21, 2006).

Dedication:

Page	Type	Explanation
v	Note	I must include my wonderful daughter, Lauren, who was born after this book was published.

General:

Page	Type	Explanation
All	Note	It is my preference in this book when expressing a decimal value that continues on past the digits that are displayed, that the last digit shown is rounded off, followed by ellipses ("…"). For example, a value of log 2 = 0.3010299956…when shown to 5 decimal places will be given as 0.30103… rather than 0.30103 or 0.30102… This helps a great deal when comparing an approximation to an exact value to a certain number of places.

Chapter 1: A Time for Reckoning

Page	Type	Explanation
-	-	No Notes or Errata.

Chapter 2: Primitives

Page	Type	Explanation
12	Note	To get a more intuitive, pictorial feel for the use of Equation 1, as well as other general multiplication/squaring methods, refer to my ***Mental Calculation Class Handout***, which is available via a link on the webpage for Chapter 2.
12	Note	To solve 37x43, Equation 1 can be used to convert this to $40^2 - 3^2 = 1591$. However, if the numbers do not have a midpoint, you can either assume there is one and subtract the difference at the end, or use another shortcut--the two numbers on either side of the midpoint are multiplied, and the product of their offsets is subtracted. So 36x43 = 39x40 – 3x4 = 1548 (see http://www.urticator.net/essay/5/592.html).
14	Error	***"125 x n = 8n,000 / 8"*** should read ***"125 x n = n,000 / 8"***
15	Error	***"111 x a||b = a | [(a+b)x11] | b"*** should read ***"111 x a||b = a | [(a+b)x11] | |b"***
15	Typo	***"37 = 111/4"*** should read ***"37 = 111/3".*** I have a funny thing there--for some reason I intuitively think that 111/37 is 4 rather than 3. I don't know why I think that.
16	Typo	***"58^2 = 3264"*** should read ***"58^2 = 3364"***

15 - 18 *	Note	A mnemonic scheme (mixed with some mental calculation techniques) for squares and cubes of two-digit numbers can be found in the paper ***Mnemonics for Squares and Cubes of Two-Digit Numbers*** on the webpage for Online Material. Consider it a fun diversion.
20	Note	Technically the period of 1/43 could also be 1, but it obviously it will not be here.
	Error	***"…are all different and are greater than 5"*** should read ***"…are all different and are greater than 3"*** as in the example that immediately follows this statement.
	Note	The factors of 9, 99, and 999 also include 27, and technically also 9, 99, and 999, which are included in the count of 63.
	Note	Just to be clear about this, corresponding digits in half periods add to 9 for any irreducible fraction s/t with t prime and an even group length, regardless of whether that length is (t-1). For example, this property holds for the primes 11 and 13. (It also holds for some composite denominators with even group lengths, such as 77.)
	Note	Consider the sentence "In another special case of reciprocal 1/t where t = 2^a x 5^b x t_1^n and t1 is any prime between 5 and 487, the period of the repeating group equals that of 1/t multiplied by t_1^{n-1}." The reason that the 486-digit repeating group for t = 487 does not double for $1/487^2$ is that the repeating group 1/487 is actually divisible by 487, so when you take 1/487 and divide that by 487 again, you end up again with a 486-digit repeating group. This is also true for t = 1/3, since 0.3… has a repeating group that is divisible by 3, so $1/3^2$ ends up with a single-digit group as well. The next example of this is 56598313, with no further ones known. John McIntosh has done an ingenious analysis to show that no other example exists below 2×10^{11}, and probably not below $5x10^{30}$ (http://www.urticator.net/essay/6/616.html).
21	Error	***"They always will if no factor of t divides 10^n - 1, for n a positive integer."*** should read ***"They always will if no factor of t divides 10^n - 1, for n = 1/2 of the group length."***
	Note	***"If t is not divisible by 2, 3, or 5 and 1/t has a period of (t-1) digits, this situation will occur as well."*** This is technically correct, and in fact is stated in this way in the source reference, but in fact it is moot because no non-prime reciprocal 1/t has a recurring period of (t-1).
22	Note	The line that calculates $-2^2/43$ should have a negative sign inside the parentheses in front of .0465…
23	Typo	***"1/13 = 0.076913"*** should read ***"1/13 = 0.076923"***
24	Error	***"... the adjusted remainder ended up greater than the divisor 80"*** should read ***"...the adjusted remainder ended up greater than the divisor 78".***
25	Typo	***"... the dividend by 10n"*** should read ***"... the dividend by 10^n"***
27	Typo	The columns can be seen to be misaligned for the 1/39 calculation.
	Typo	The 3042 at the end of the 1/13 calculation should be 3072.
35	Typo	***"50/7 = 7.142286"*** should read ***"50/7 = 7.142857"***
37	Note	In the table for the modified Euclid 's algorithm, 441 = 11x35 + 8x7 could also be represented by 441 = 13x35 - 2x7.
44	Typo	114 in the next-to-last line in Table 1 should be 1140.
45	Error	57 is not prime, so the combination divisibility test for 47, 53 and 57, while correct as given, is actually a divisibility test for 3, 19, 47 and 53 if we are interested in prime divisibility. Note that if we test for 47x53x59=146969, the product is almost as convenient to work with as it was for 47x53x57=141987.
46	Typo	***"the prime numbers 3 and 9"*** should read ***"the numbers 3 and 9"***
47	Note	We can add more entries to Table 2: base 80 can have an elevens-test factor of 27, base 100 can have a nines-test factor of 33, and base 1000 can have a nines-test factor of 27 and elevens-test factors of 77 and 91.
48	Typo	Omit ***"obviously not greater than N^1/2"*** at the end of the sentence containing ***"the largest factor (prime or not) of N"***
49	Typo	The formula for number of steps in Fermat's factoring method should read ***"$(1-k)^2 * N^{1/2}$ / 2k"***. This is correctly given on page 73.
	Error	***"A square ending in 25 can only end in 125, 225, or 625"*** should read ***"A square ending in 25 can only end in 025, 225, or 625"***.

53	Typo	***"[(52)^2 - N]"*** should read ***"[(54)^2 - N]".***
	Typo	***"the next possible value after x=42 would be 114"*** should read ***"the next possible value after x=42 would rise further to 138".***
	Error	***"A case where a/b is approximately 3/2, of course, would benefit from setting k=6, the case a/b approximately 5/3 by letting k=15, and so forth."*** should read: ***"In order to get an integer midpoint of the two factors when a is not an even multiple of b, we either have to multiply by an odd number with factors of about the right ratio, or multiply by a number with two even factors of about the right ratio. A case where a/b is approximately 3/2 would benefit from setting k=24, since 24=6x4 and 6/4=3/2. When a/b is approximately 5/3 (or even 3/2), we can set k=15=5x3, and so forth."***
55	Typo	The a^5+b^5 factorization should have a plus sign on a^2b^2 term.
60	Error	In Table 4, there are additional values of tx mod m for m=4, 6 and 8, since t(t+1)/2 is not relatively prime to an even m, as shown in the table at the end of this document. The net result is that the cases m=4 and m=8 do not limit possible values of x mod m at all, and are not useful as sieves. The case m=6 retains the same limitations as before, but is redundant to the m=3 sieve, and both of them are inferior to the m=9 sieve. So the triangular modular seives for even m are not useful.
62	Typo	***"increase the adder"*** above the N-x^2 equation should read ***"decrease the adder"***.
67	Error	All values of r should be calculated to make sure no squares appear (here we should add r2=401 and r3=201).
68	Note	To decrease the number of possible values of x to roughly 521, a mod 8 sieve was also performed.
71	Error	***"For x=605, we know that (x^2 - N) mod 9 = [(2)^2 - 4] mod 9 = 0; because of the earlier x mod 9 sieve, all of these values of x will give this result. Therefore, only y=234 is possible, since the nines test on the ending 34 implies an initial digit of 2 for the range 0<y<505. An ending of 66 produces no initial digit in this range. As y=234 seems a likely possibility, we cast out elevens to check it again, producing (x^2 - N) mod 11 = [(0)^2 - 7] mod 11 = 5. The y-value 234 is now eliminated because (234)^2 mod 11 = 9; thus x cannot be 605. For x=515, again only y=234 is possible and is eliminated by the elevens test. For x=425, we find y=234 does indeed pass the elevens test. For x=565, we find that y=234 passes the elevens test as well."*** should read: ***"For x=605, y must be less than 505. The mod 9 sieve acting on y-endings of 34 limits these to 234, which also passes the mod 7 sieve. The mod 9 sieve acting on y-endings of 66 in this range limits these to 66 and 366, of which the mod 7 sieve leaves only 66. T he elevens test gives (x^2 - N) mod 11 = [(0)^2 - 7] mod 11 = 4. The y-values 234 and 66 are now eliminated because (234)^2 mod 11 = 9 and (66)^2 mod 11 = 0. Without any possible y, x cannot be 605. For x=515, again y=234 and y=66 are eliminated by the elevens test. For x=425, we find y=234 does indeed pass the elevens test, but 66 does not. For x=565, we find that y=234 passes the elevens test as well, but 66 does not."***
72	Typo	In the next-to-last equation line, 367 should read 361.

Chapter 3: Roots

Page	Type	Explanation
79	Typo	66577993506607 should read 6657793506607. This does not affect the result.
80,85	Typo	7.14142842857… should read 7.14142842856… This does not affect any result.
82	Typo	The squared term in the denominator of Equation 7 should not be squared. This does not affect anything else.
83	Note	John McIntosh derives Aitken's fourth square root method by a different means. If the midpoint of a and N/a is M, and the deviation of M from a or N/a is D, then N = a x N/a = (M-D)(M+D) = $M^2 – D^2$. Then $N^{1/2} = (M^2 – D^2)^{1/2}$ = M x $(1 – (D/M)^2)^{1/2}$. If we assume that D/M is small and take the first term of the Taylor series, we get $N^{1/2}$ = M x (1 –

		$(D/M)^2/2$). The value D/M is the fractional deviation that Aitken refers to, and this formula tells us to reduce M by half the deviation squared, which is what Aitken does in his fourth estimate.
84	Note	To avoid any possible confusion between the prime symbol used to represent an assignment, in this case $a_n' = N/a_n^{p-1}$, and the prime symbol used to represent the solution of Halley's method, you might think of the last two general equations beginning with $f(a_n)'$ rather than a_{n+1}'. The method is used correctly in the text.
	Typo	In the bottom equation, a_n should read a_1 since n=1, and the result should be 7.141442715700.
85	Typo	In the formula for a_2, 49x50 in the denominator should read 49x51. The result is correct.
89	Note	An alternate, more intuitive derivation of the general square root algorithm can be found in the paper ***An Alternate Derivation of the General Square Root Algorithm*** on the webpage for Chapter 3.
	Typo	In the bottom equation, a_n should read a_0 as it does in the following line.
90	Typo	In the top equation, ***"-2n-2"*** should read ***"-2n-4"*** as it does when that term next appears (at the top of page 91). This does not affect the result.
91-92	Typo	**"+ R_{n+1}"** at the end of the equations at the bottom of page 91 and the top of page 92 should read **"- R_{n+1}/a_0"** to be consistent everywhere.
93	Note	In the square root example, b4=54 is actually greater than 50 despite my preference for keeping b's less than 50. This does not invalidate the example, as the reasons for limiting the size of b are to ease multiplications and to limit ripple effects to neighboring number groups. In this case the value was very near 50. The same example done with all b's kept less than or equal to 50 is found in the paper ***Square Root With Low Number Groups*** on the webpage for Chapter 3.
94	Note	Other examples of finding a square root with the general algorithm from this chapter, and ones with more explanation of the steps involved, can be found in the papers ***Another Square Root Example*** and ***Yet Another Square Root Example*** on the webpage for Chapter 3.
97	Note	I feel a little guilty about saying that the approximation for the reciprocal square root contains no division (although other authors have no hesitation in doing so), when the 0.5 multiplier is in practical terms a division by 2.
98	Typo	The result .1400280084023… should read .1400280084025…, which is even more stunningly accurate than the result given.
	Note	The approximation of 1/(1-.0003) by 1+.0003 uses the first two terms of the Binomial Series expansion for $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$ This provides an accuracy of .1400280084. The full accuracy of .14002800840025… can also be obtained quite easily by including the third term $(-.0003)2 = 9x10^{-8}$. Multiplying this by $2.8x10^{-5}$ adds $2.52x10^{-13}$ to achieve the higher accuracy without much effort.
100	Typo	The result 4.91875… should read 4.91876...
	Typo	In the first equation, a_0 in the denominator inside the parentheses should read a_0^{p-1}. The example given is correct.
	Note	The Chebyshev correction term to subtract for cube roots can also be calculated as $(a_1-a_0)^2/a_0$, which is simpler here since $(a_1-a_0) = -.08$ is a single digit in this example. See the last section on cube roots in the paper ***An Alternate Derivation of the General Square Root Algorithm*** on the webpage for Chapter 3.
	Typo	In the last line of text, a_{n+1} should read a_{n+1}'. This does not affect the derivation.
101	Typo	In the equation above "by the Binomial Theorem", the second term after the division sign should be positive rather than negative. In the text in the middle of the page, "8_p^2" should read "$8p^2$". The overall derivation is correct.
104	Typo	4.4979408… should read 4.4979414, which makes the cube root approximation even more accurate.
106	Typo	To be consistent on rounding the digit before the ellipsis (…), 4.7328… should read 4.7329…
107	Typo	5.5349164… should read $5.5349164^{1/2}$. The result of 2.16894364… should read 2.16894354… which now matches the desired $48^{1/5}$ to the last digit calculated.

Chapter 4: Logarithms and Their Inverses

Page	Type	Explanation
115	Note	To be clear, the value of x in the equation halfway down the page would be N.
116	Typo	In Equation 14, the range should read $-1 < x <= 1$.
117	Typo	In the last equation on the page for log(1+x), the range should read $10^{-m} < x < 10^{-m+1}$.
122	Typo	2/2423 should read 2/2421 since N=1210. This is not sufficient to affect the result to the number of digits taken.
	Note	It may seem that 2 log 11 + log 10 = 3.08279 should be 3.08278 if we use the value given to five decimal places for log 11 on page 119 (1.04139). However, the value of log 11 to 6 decimal places is 1.041393…, so 2 log 11 actually rounds to 2.08279. The purpose on page 122 was to find the ultimate attainable precision of the formula, so the number of digits used for log 11 was not limited.
122-123	Error	In Equation 16, and again on page 123, the term 3(2N+1) should be squared. The term 3(2N+a) should be also be squared on page 123.
123	Error	In the general equation for ln(N+a), the 2a/(2N+a) should be added, not subtracted, and the numerators of the last two terms should be a^2 and a^4, respectively.
124	Error	In the top equation on this page, the signs in front of the each "½" on the right-hand side of the equation should be reversed.
125	Typo	In the top equation, only one side should have a negative sign.
126-127	Typo	In the second equation from the bottom on page 126 and the second equation from the top on page 127, the term 2d should be simply d. This does not affect the validity of the ultimate Equation 18.
128	Typo	***"if we actually knew log 22 and log 23"*** should read ***"if we actually knew log 22 and log 24".***
130	Note	The final result 1.1139589 is shown to more digits than the intermediate results in order to show more clearly the accuracy of the formula.
131	Note	Just to avoid possible confusion, the values a and g are not exactly the same as the arithmetic-geometric mean. The equations show the correct meanings of these.
132	Typo	The equation $N'=10^p2^mN$ should read $N=10^p2^mN'$. There is no effect on any results.
135	Typo	In the equation for d_3, the numerator is reversed. The result is correct.
136	Typo	In Table 8 the error for 4/3 should be $2x10^{-4}$.
140	Typo	In the two lines above the result 47.00, a factor of 11/10 is not shown. The result is correct.
	Typo	***"b<=.005"*** should read ***"|b|<=.005"***.
137 – 142 *	Note	Tips to lessen the effort when using the Bemer method can be found in the paper ***The Practical Use of the Bemer Method for Exponentials*** on the webpage for Chapter 4.
	Note	John McIntosh presents an alternative to the Bemer method (the McIntosh-Doerfler method) for mentally calculating exponentials in his essay ***Exponentials*** at http://www.urticator.net/essay/6/641.html

Chapter 5: Trigonometric Functions and Their Inverses

Page	Type	Explanation
147	Note	The value 174 in Equation 22 is different from the rounded value of 174.51 from the previous equation because 174 provides a better approximation over the whole range. The adjustment of this value for overall accuracy is also done in Equation 23 and Equation 27.
148	Typo	The initial slope of the sine function should be given as 0.017453.
151-152	Typo	I say that the sine approximation is valid in the range where the cosine approximation is invalid, and vice-versa. The next two sentences show what I really meant there—that the range of angles in which the sine approximation is valid (0-54 degrees) and the

		range of angles in which the cosine approximation is valid (0-36 degrees) are convenient because for a>54, we can replace sin(a) with cos(90-a) and watch the sign on the answer, and for b>36, we can replace cos(b) with sin(90-b).
152	Note	In the equation that merges Equations 23 and 24, the coefficient would actually be 171.43, but is given as 172 to match the formula given by Ozanam.
153	Note	The value 0.53168 for tan 28º would be 0.53166 if the sine and cosine values are taken only to the 4 decimal places given in the earlier calculations.
154	Typo	The reference to Equation 23 should be to Equation 22.
156	Typo	In the top equation, ***“28x30/40”*** should read ***“28x30/20”***. The result is correct.
	Note	To get the result 0.53145 for tan 28º, tan 30º must be known to 5 decimal places. The intent here is to find the ultimate accuracy possible from the formula.
157	Typo	In Equation 33, the sign on .5 should be reversed.
	Note	The caption for Figure 6 should indicate that the sign of the y-value is reversed for the plot from Equation 33 to show the intersection of the absolute values of the errors of the two plots.
158	Typo	The phrase ***“dropping a*** b^4 ***term”*** should read ***“dropping a*** b^3 ***term”***.
160	Note	The caption for Figure 8 should indicate that the sign of the y-value is reversed for the plot optimized for x<=.707 to show the intersection of the absolute values of the errors of the two plots.
161	Typo	One of the g subscripts is a letter “o” instead of a number “0”.
165	Typo	The arctanh formula should use ln.

Chapter 6: Concluding Remarks

Page	Type	Explanation
-	-	No Notes or Errata.

Appendix: Finding Rational Approximations to Precomputed Constants

Page	Type	Explanation
All *	Note	A graphical software program for generating rational approximations to a decimal number entered by the user can be found on the webpage for the Appendix.
172	Typo	In the top paragraph, 1/2n in $10^{1/2n}$ should read $1/2^n$.
173	Typo	The phrase ***“*** a_0 ***greater than 1”*** should read ***“*** a_n ***greater than 1”***.
176	Note	To be consistent with my general philosophy of rounding the last digit even when an ellipsis (…) exists at the end, the value 1.1555555… should read 1.1555556…

Update to Chapter 2, pg. 60, Table 4 (see errata above)
Modular Sieves Table for Triangular Number Factoring
—Even M's Only

m	x	t_x	t_x mod m	N mod m	$(t_x – N)$ mod m	Possible x mod m
4	0, 1, 2, 3, 4, 5, 6, 7	0, 1, 3, 6, 10, 15, 21, 28	0, 1, 3, 2, 2, 3, 1, 0	1	Irrelevant—since all possible t_x mod m appear (0-3), any x mod m is possible (0-3).	
				3		
6	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66	0, 1, 3, 0, 4, 3, 3, 4, 0, 3, 1, 0	1	5, 0, 2, 5, 3, 2, 2, 3, 5, 2, 0, 5	1, 4, 7, 10 → 1, 4
				5	1, 2, 4, 1, 5, 4, 4, 5, 1, 4, 2, 1	0, 2, 3, 5, 6, 8, 9, 11 → 0, 2, 3, 5
8	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15	0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120	0, 1, 3, 6, 2, 7, 5, 4, 4, 5, 7, 2, 6, 3, 1, 0	1	Irrelevant—since all possible t_x mod m appear (0-7), any x mod m is possible (0-7).	
				3		
				5		
				7		

Creating the last column of a given row can be confusing. The way I look at it is that I am finding which positions in the cell for $(t_x – N)$ mod m have numbers that appear also in the cell for t_x mod m.