Solution to *Edwards' Challenge*

A Mathematical Puzzle by Ron Doerfler [1]
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The solution to the puzzle is presented here, first as a list of answers and then as a solution with detailed explanations.

Answer List

1. Minimum value of N at this point in the puzzle is 0000000000000222 (or just 222)
2. Maximum value of N at this point in the puzzle is 8888866688888888
3. Only possible sum of odd-place digits of N is 24
4. Only possible sum of even-place digits of N is 46
5. Minimum value of N is 0060828800828488 (or just 60828800828488)
6. Maximum value of N is 2248808088846220

Detailed Solution

There are many ways to attack this problem. This detailed solution provides one way to solve the problem to show that it is possible.

We are given a 16-digit number N with all digits even and leading zeros allowed. Even place digits in the upper half appear in even places in the lower half, except one in the lower half is increased by 2. Odd place digits in the upper half appear in odd places in the lower half, except two in the lower half are increased by 2 each.

The first two answers are simply deduced from these constraints:

1. The minimum value of N so far is 0000000000000222 (or just 222).
2. The maximum value of N so far is 8888866688888888.

Now, $N/9$ leaves a remainder of 7, and $N/11$ leaves no remainder. From this we can deduce from simple trial and error that $N/99$ leaves a remainder of 88, since 9 and 11 are factors of 99, and $88/9$ leaves a remainder of 7 and $88/11$ leaves no remainder. This is written succinctly as

$$N \bmod 9 = 7$$
$$N \bmod 11 = 0$$
$$N \bmod 99 = 88$$

[1] The puzzle *Edwards' Challenge* can be found at *http://www.myreckonings.com/Dead_Reckoning/Online/EdwardsChallenge.pdf*

Let N be given as

$$N = abcdefghijklmnop$$

where the lower-case letters represent digits. Now, divisibility by 99 is equivalent to summing pairs of digits, for the same reason that divisibility by 9 is equivalent to summing individual digits (one less than base 100 in the first case and one less than base 10 in the second case). Therefore,

$$ab + cd + ef + gh + ij + kl + mn + op = 88 \text{ more than a multiple of } 99$$

Obviously, the maximum that 8 pairs of even digits can be is $8 \times 88 = 704$. So, the possibilities are 88, 187, 286, 385, 484, 583, and 682. We know that all the digits are even, though, so we know the answer is even, narrowing the list to 88, 286, 484, and 682. In addition, we know that

$$(b + d + f + h) + 4 = (j + l + n + p)$$
$$(a + c + e + g) + 2 = (i + k + m + o)$$

Since the even digits are tens places in the overall equation, the adder 2 becomes an adder of 20, which is divisible by 4. Therefore, the overall sum must be divisible by 4, so we are left with two possibilities:

$$ab + cd + ef + gh + ij + kl + mn + op = 88 \text{ or } 484$$

Now we know that the sum of the odd-place digits ends in 4 or 8 from these results. Since the odd-place digits consist of pairs of equal digits except for an increase of 4, then the sum must be divisible by 4 and have a range from 4 to roughly $8 \times 8 - 4 = 60$. The possible sums of odd-place digits are then 4, 8, 24, 28, 44, and 48. Let's create a table of what we know so far:

Sum of Pairs	*Sum of Odd-Place Digits*
88	8
	28
	48
484	4
	24
	44

Possible Sums of Digits of N so far.

We can fill in a column for the *Sum of Even-Place Digits* by looking at the other two columns and recalling that the even-place digits are the tens in the addition of pairs. For example, for a *Sum of Pairs* equal to 88 and *Sum of Odd-Place Digits* equal to 28, the tens, or *Sum of Even-place Pairs*, must be 6, since $60 + 28 = 88$. Knowing the *Sum of Odd-Place* and *Sum of Even-Place Digits* allows us to total the *Sum of All Digits*. We find:

Sum of Pairs	*Sum of Odd-Place Digits*	*Sum of Even-Place Digits*	*Sum of All Digits*
88	8	8	16
	28	6	34
	48	4	52
484	4	48	52
	24	46	70
	44	44	88

Possible Sums of Digits of N.

Now we need to look further to reduce these possibilities.

Divisibility by 11 is equivalent to subtracting even-place digits from odd-place digits. Therefore, since $N \bmod 11 = 0$,

$$(b+d+f+h+j+l+n+p)-(a+c+e+g+i+k+m+o) = \text{a multiple of } 11 \tag{1}$$

The range of this multiple is roughly (-8×8) to 8×8, or -64 to 64. Also, we know the result must be even, since all the digits are even. This narrows the possibilities to $-44, -22, 0, 22$, and 44. If we realize that the even-place digits are duplicated except for an increase of 2, and the odd-place digits are duplicated except for an increase of 4, we see that the possibilities must not be divisible by 4, so we are left with possibilities of -22 and 22 for the difference between the odd-place and even-place digits in N. This eliminates rows 1, 3, 4, and 6 above, leaving

Sum of Pairs	*Sum of Odd-Place Digits*	*Sum of Even-Place Digits*	*Sum of All Digits*
88	28	6	34
484	24	46	70

Remaining Possible Sums of Digits of N.

Divisibility by 9 is equivalent to adding all the digits of N. The last column above does indeed leave a remainder of 7 when divided by 9. However, since

$$\begin{aligned}\text{Sum of All Digits} &= 2(\text{sum of 8 even numbers}) + 2 + 4\\ &= 16(\text{even numbers}) + 6\end{aligned}$$

then we see that $70 = 16\times4+6$, which fits, but 34 does not. Therefore, only the last entry of the most recent table survives, and we know the sums:

3. The only possible sum of odd-place digits is 24.
4. The only possible sum of even-place digits is 46.

Now the above answers mean that for the odd-place digits, the upper half add to 10 and the lower half to 14, since two of the latter digits are increased by 2 apiece. Also, for the even digits, the upper half add to 22 and the lower half to 24, since one of the latter digits is increased by 2.

$$\begin{aligned}b+d+f+h &= 10\\ j+l+n+p &= 14\\ a+c+e+g &= 22\\ i+k+m+o &= 24\end{aligned}$$

Realizing the constraints on digits in the two halves, it becomes apparent that for minimum N, we want to push high digit values toward the right as far as possible. We get

$$\begin{aligned}&bdfh = 0028 \qquad jlnp = 0248 \qquad \text{(two digits in right half increased)}\\ &aceg = 0688 \qquad ikmo = 0888 \qquad \text{(one digit in right half increased)}\end{aligned}$$

giving:

5. The minimum value that N can be is 0060828800828488.

We may be tempted to turn the digit patterns around to push high values to the left to get a maximum N of 8882600088848200. However, (and this is *tricky*), Edward said that N was expressed as a 17-digit octal value and as a 15-digit duodecimal value. The maximum value of these are $8^{17} = 2251799813685248$ and $12^{15} = 15407021574586368$. (These can be calculated by hand—that's how I did it originally.) The smallest of these is 8^{17}, which sets a limit on the largest value that N can be. Adjusting the digits above to find a maximum below this value, we find

$$bdfh = 2800 \qquad jlnp = 8420$$
$$aceg = 2488 \qquad ikmo = 8862$$

so,

6. The maximum value that N can be is 2248808088846220.