Mnemonics for Squares and Cubes of Two-Digit Numbers

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As a tool for mental calculation, I have never been particularly fond of mnenomics, that is, methods for memorizing numbers by word association or letter patterns. Perhaps I would be if I were a performer called upon regularly to recall numbers or items, but my interests lie more along the lines of calculations one might encounter in the course of daily life. Since these can be varied and infrequent, I have a bent toward general rather than particular methods. That said, I found a set of mnemonics from a book published in 1910 to be a refreshing and fun way to quickly call to mind cubes of two-digit numbers, which is a sufficiently common task in the general methods to warrant an exception. The book is titled *Magician's Tricks: How They are Done*, by Henry Hatton and Adrian Plate, and the chapter that contains their mnemonic system can be found at http://stepanov.lk.net/mnemo/mgtr.html. In this paper, I have extended their scheme to provide squares as well as cubes of two-digit numbers, as these are so important for mental calculation. In addition, I have updated a number of their mnemonic phrases to ones using more modern terms, or to ones I think are an improvement over the original phrases.

Why Squaring is So Important in Mental Calculation

The ability to quickly find the square of a number is critical for fast mental calculation. Consider these applications:

1. The straightforward calculation of a square is a common task in mental arithmetic.

2. One of the most powerful tools in mental calculation is converting the multiplication of two different numbers into the square of the average minus the square of the difference. This is shown by the algebraic identity:

$$(a+c)(a-c) = a^2 - c^2$$

   where **a** is the average of the two numbers, **(a+c)** is one of the numbers, and **(a-c)** is the other number. This is a very common technique in mental calculation. For example, 28x32 = $30^2 - 2^2$, and 53x77 = $65^2 - 12^2$. Less convenient multipliers can be manipulated in a number of ways to use this technique. Here are a couple of examples:

$$\begin{aligned} 28\text{x}33 &= 28\text{x}32 + 28 = 30^2 - 2^2 + 28 \\ \text{or} &= 28\text{x}34 - 28 = 31^2 - 3^2 - 28 \\ 23\text{x}67 &= 23(100\text{-}33) = 2300 - 23\text{x}33 = 2300 - (28^2 - 5^2) \\ \text{or} &= 23\text{x}33\text{x}2 + 23 = 2(28^2 - 5^2) + 23 \\ \text{or} &= 23(50+17) = 2300/2 + 23\text{x}17 = 2300/2 + 20^2 - 3^2 \end{aligned}$$

   In the end, we reduce all the possible combinations of two-digit by two-digit multiplications to just the two-digit squares plus minor arithmetic adjustments.

3. Multiplication of two three-digit numbers can be converted as above: $244x376 = 310^2 - 66^2$. But 310^2 is really just a two-digit square followed by two zeros—what if we end up with a three-digit square here? Then we use the identity in #2 above *in reverse*, so we split the square into two numbers equidistant from the original number, *adding* the square of that distance. Consider $244x382 = 313^2 – 69^2 = [300(326) + 13^2] – 69^2$ and we end up with a simple calculation if we know the two-digit squares. Remember, the average squared is greater than the split numbers multiplied, so if you square the average of two numbers, you *subtract* the difference squared; if you split a square into two equidistant numbers multiplied, you *add* the difference squared. Squares of three-digit or four-digit numbers can use the expansion $(a+b)^2 = a^2 + b^2 + 2ab$, where the 2ab multiplication can again be simplified using squares. If the number has three digits, the middle digit can be included in either **a** or **b**, depending on which squares the calculator is most comfortable using.

4. Calculating square roots involves finding squares of intermediate solutions. This is true not only for the classic square root method, but also for alternatives based on Newton's method or other methods more suited to mental calculation, as described in my book, *Dead Reckoning: Calculating Without Instruments*. Calculating cube roots by taking weighted averages, described in the book, also requires squaring operations.

5. Power series formulas for functions such as logarithms and exponentials involve raising numbers to powers. For mental calculation, terminating the series at a power of 2 usually provides satisfactory precision. One example of using a power series that includes the squared term can be found in my paper on mentally calculating exponentials at http://www.myreckonings.com/Dead_Reckoning/Chapter_4/Materials/Bemer_Exponentials.pdf

These considerations have led me to revise my original opinion of mnemonics. I think mnemonics can be useful to a mental calculator as a fun tool for learning squares and as a more permanent method for producing cubes. However, there is no good reason to remember mnemonics for squares that are easily found in other ways, and there are very good reasons to learn the general strategies. Therefore,

- There are no entries in the table for multiples of 10, since $30^2 = 3^2x100$ and $30^3 = 3^3x1000$.

- Squares of numbers in the ranges 11-22 and 91-99, and all numbers ending in 5, are listed with a shortcut formula based on the reverse of the identity in #2 above. We split the square into two numbers each equidistant from the original number, multiply them, and then add the distance squared. The square of 13, for example, can be converted to $10x16 + 3^2$, which is easily calculated. Above 15 we can use either 10 or 20 as the convenient multiplier: $18^2 = 10x26 + 8^2$ or $16x20 + 2^2$, but I prefer using 20. We really could have extended this to 29, and indeed all the squares can be split to a nearby multiple of 10, but the mnemonics can also be used here. The squares of numbers in the 90's all use 100 as a convenient multiplier. Squares of numbers ending in 5 are split into the multiples of 10 on either side, with 5^2 added—since both multipliers end in 0, we can just multiply the first digit by the first digit plus one, then append 25: $65^2 = (6x7) | 25 = 4225$.

- Squares of numbers in the range 41-59 are listed with a different shortcut formula. For numbers near 50, we can add the difference from 50 to 25, multiply by 100, and add the difference squared. Since the distance is within 10 in the range 41-59, we can add the difference to 25 and simply append the distance squared. Here $53^2 = (25+3) | 3^2 = 2809$ and $44^2 = (25-6) | 6^2 = 1936$.

The Mnemonic Alphabet for Digits

This mnemonic scheme uses consonant sounds to replace digits according to the table below.

The way to initially remember this table is:

- t has one down-stroke
- n has two down-strokes
- m has three down-strokes
- r is the last letter of four
- L in Roman notation is fifty
- J looks something like a reversed six
- k, inverted, is similar to seven
- f in script resembles eight
- p is similar to a flipped nine
- c is the first letter of cipher, which is the word for naught.

Digit	Consonant Sound
1	t, d, th
2	n
3	m
4	r
5	l
6	j, ch, sh, zh, z as in azure, soft g as in genius
7	k, hard c, q, hard g, ing
8	f, v
9	p, b
0	s, z, soft c

It is important to treat these as sounds, not letters—this will help you remember the rest of the entries in the table.

The vowels a, e, i, o and u, and w, h, y are merely used to form words, as nail (n-l) = 25 and chess (ch-s) = 60. Silent letters, those that are not pronounced, have no value, as for example, knife (n-f) = 28; lamb (l-m)= 53; gh in thought; l in palm; and r in tapeworm. Double consonants are treated as one letter, as mummy (m-m) = 33 and butter (b-t-r) = 914, but if the double letters have distinct articulation, then each letter has its own numerical value, as accept (k-s-p-t) = 7091 and bookkeeper (b-k-k-p-r) = 97794. Since a whole number doesn't begin with zero, the letter s may be used at the beginning of the mnemonic to form a better phrase.

The idea here is that the keyword for the two-digit number will come to mind from the alphabet, and then the phrase associated with that keyword for the square or cube will be pictured, and then the result read out left-to-right from the alphabet sounds of the phrase.

Note that this mnemonic alphabet is identical to that presented as an aid for remembering intermediate results of mental calculations in the excellent books by Arthur Benjamin. However, the reference given earlier in this paper to the book published in 1910 makes it clear that this particular scheme was used at least as far back as the famous magician Harry Kellar.

So now we begin. The first column below is the original number, the second column is the square of the number, and the third column is the cube of the number. The starting numbers for the 1,000's, 10,000's, and 100,000's are noted in the table so that the proper grouping can be spoken as the number is read out from left to right. Oh, and please email me at ron@myreckonings.com if you come up with any better mnemonic phrases for a newer version of this paper—I'd love to hear them.

Mnemonics for Squares and Cubes of Two-Digit Numbers

3 Ham	9	2 7 An egg	
4 Rye	16	6 4 Sherry	
5 Lie	25	1 2 5 Denial	
6 Hash	36	2 1 6 On a dish	
7 Key	49	3 4 3 My room	
8 Hive	64	51 2 Wild honey	
9 Bee	81	7 2 9 Go nip	
11 Date	10x12 + 1^2 = 121	1 3 3 1 With my mate	___ Begin 1,000's for cubes at 11
12 Dine	10x14 + 2^2 = 144	1 7 2 8 Take enough	
13 Item	10x16 + 3^2 = 169	2 1 9 7 Notebook	
14 Author	10x18 + 4^2 = 196	2 7 4 4 Ink hirer	
15 Tell	(1x2)|25 = 225	3 3 75 Me meekly	
16 Ditch	20x12 + 4^2 = 256	4 0 9 6 Rosebush	
17 Talk	20x14 + 3^2 = 289	4 9 1 3 Rap time	
18 Thief	20x16 + 2^2 = 324	5 8 3 2 I love money	
19 Daub	20x18 + 1^2 = 361	6 8 59 Shave a lip	
21 Hand	20x22 + 1^2 = 441	9 2 6 1 Punched	
22 Nun	20x24 + 2^2 = 484	1 0 6 4 8 Does show her vow	___ Begin 10,000's for cubes at 22
23 Name	5 2 9 Will nab	1 2 1 6 7 A dandy joke	
24 New Year	5 7 6 Lucky age	1 3 8 2 4 With my fine rye	
25 Nile	(2x3)|25 = 625	1 5 6 2 5 The lush Nile	
26 Wench	6 7 6 Watch cash	1 7 5 7 6 Took all cash	
27 Nag	7 2 9 Go nap	1 9 6 8 3 To buy each wife a home	
28 Enough	7 8 4 Giver	2 1 95 2 Neat plan	
29 Nip	8 4 1 Ferret	2 43 8 9 In arm of boy	
31 Mad	9 6 1 Pushed	2 9 7 9 1 In a big pout	
32 Man	1 0 2 4 Thy sinner	3 2 7 6 8 Many catch a wife	___ Begin 1,000's for squares at 32
33 My Home	1 0 8 9 Do save up	3 5 9 3 7 Home will be a hammock	
	1 1 5 6	3 9 3 0 4	

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34 More	Do tell a wish	May be a miser 4 2 8 75
35 Mall	(3x4)|25 = 1225 1 2 9 6	Hurry in a vehicle 4 6 6 5 6
36 Smash	Stone bash 1 3 6 9	Rich jewel show 5 0 6 5 3
37 Smoke	Dumb chap 1 4 4 4	Lose a chilly home 5 4 8 7 2
38 Move	Steer rear 1 5 21	Lower a heavy can 59 3 1 9
39 My pay	To lend	Help me to buy 6 8 9 21
41 Road	$(25-9)\|00 + 9^2 = 1681$	Chief point 74 0 8 8
42 Run	$(25-8)\|00 + 8^2 = 1764$	Across a five 7 9 5 0 7
43 Rome	$(25-7)\|00 + 7^2 = 1849$	Keep losing 8 5 1 8 4
44 Rower	$(25-6)\|00 + 6^2 = 1936$	Awful diver 9 1 1 2 5
45 Rail	(4x5)|25 = 2025	A bad tunnel 9 7 3 3 6
46 Rich	$(25-4)\|00 + 4^2 = 2116$	Big mummy show 10 3 8 2 3
47 Rake	$(25-3)\|00 + 3^2 = 2209$	Hits my wife numb 1 1 0 5 9 2
48 Rough	$(25-2)\|00 + 2^2 = 2304$	The wide sea will open 1 1 7 6 4 9
49 Rope	$(25-1)\|00 + 1^2 = 2401$	To take a chair up 1 3 2 6 51
51 Lad	$(25+1)\|00 + 1^2 = 2601$	The man child 1 4 0 6 0 8
52 Lion	$(25+2)\|00 + 2^2 = 2704$	Dares chase a foe 1 4 8 8 7 7
53 Lamb	$(25+3)\|00 + 3^2 = 2809$	Drove off a cook 1 5 74 6 4
54 Liar	$(25+4)\|00 + 4^2 = 2916$	Idol crusher 1 6 6 3 7 5
55 Lily	(5x6)|25 = 3025	Dutch show my equal 1 7 5 6 1 6
56 Latch	$(25+6)\|00 + 6^2 = 3136$	Tack a latch to a hatch 1 8 5 1 9 3
57 Look	$(25+7)\|00 + 7^2 = 3249$	The evil tapeworm 1 95 1 1 2
58 Loaf	$(25+8)\|00 + 8^2 = 3364$	Double the weight now 2 0 5 3 7 9
59 Lube	$(25+9)\|00 + 9^2 = 3481$ 3 7 2 1	When I sell my coupe 2 2 6 9 8 1
61 Shed	My wagon hut 3 8 4 4	An inch by a foot 2 3 8 3 2 8
62 Jane	My fairer 3 9 6 9	Name of my new wife 2 5 0 0 47
63 Gym	Wimpy shape 4 0 9 6	Any loss is work 2 6 2 14 4
64 Chair	Why raise? Push	Now a china drawer 2 7 4 6 2 5
65 Shallow	(6x7)|25 = 4225 4 3 5 6	Niagara channel 2 8 7 4 9 6
66 Judge	Harm wily witch 4 4 8 9	No fake rubbish 3 0 0 7 6 3
67 Joke	Rare fib 4 6 2 4	Amuse a sick chum 3 14 4 3 2

___ Begin 100,000's for cubes at 47

68 Shave	Your chin raw	May try your man
	4 7 6 1	3 2 8 50 9
69 Shop	Here, cash it	Woman feels happy
	5 0 4 1	3 57 9 1 1
71 Cat	Loose-eared	Milk by the day
	5 1 8 4	3 7 3 2 4 8
72 Coin	Will die for	Make men rave
	5 3 2 9	3 8 9 01 7
73 Game	Wily man up	Move past GO
	5 47 6	4 0 5 2 2 4
74 Choir	Larkish	Rose Hill nunnery
		4 2 1 8 7 5
75 Kill	(7x8)|25 = 5625	Run, thief—I kill!
	5 7 7 6	4 3 8 9 7 6
76 Cage	Lock cage	Roomy if ape cage
	5 9 2 9	4 5 6 5 3 3
77 Cook	Sell pan pie	Relish a lime ham
	6 0 8 4	4 74 5 5 2
78 Give	Choose for	Here, a cruel loan
	6 2 4 1	4 9 3 0 3 9
79 Cop	Shine red	Rip a museum up
	6 5 6 1	5 3 1 4 4 1
81 Food	Chill shad	Well-made rare tea
	6 7 2 4	5 5 1 3 6 8
82 Fan	Watch a hockey winner	Loyal to my Chevy
	6 8 8 9	5 7 1 7 8 7
83 Foam	Shave if happy	Liquid quaffing
	7 0 5 6	5 9 2 7 0 4
84 Fire	Goes hellish	We'll ban gas here
		6 1 4 1 2 5
85 Fall	(8x9)|25 = 7225	Shatter the new well
	7 3 9 6	6 3 6 0 5 6
86 Fish	Camp chow	Wish me a juicy leech
	75 6 9	6 5 8 50 3
87 Fake	Kluge by	Jello fools me
	7 7 4 4	6 8 1 4 7 2
88 Five	Gang warrior	Shaved a raccoon
	7 9 2 1	7 0 4 9 6 9
89 VIP	Cabinet	Guess our bishop
		75 3 5 7 1
91 Piety	$100x82 + 9^2 = 8281$	Gloomy lookout
		7 7 8 6 8 8
92 Pony	$100x84 + 8^2 = 8464$	Kick a fish off a hive
		8 0 4 3 5 7
93 Poem	$100x86 + 7^2 = 8649$	"Wi-Fi" is rhyme-like
		8 3 0 5 8 4
94 Bar	$100x88 + 6^2 = 8836$	Famous loafer
		8 5 7 3 7 5
95 Pill	(9x10)|25 = 9025	Vile chemical
		8 8 4 7 3 6
96 Patch	$100x92 + 4^2 = 9216$	Favoring a match
		9 1 2 6 7 3
97 Pack	$100x94 + 3^2 = 9409$	Beaten each game
		9 41 1 9 2
98 Puff	$100x96 + 2^2 = 9604$	Part the bun
		9 7 0 2 9 9
99 Pup	$100x98 + 1^2 = 9801$	Big sunny puppy