Developing a "Number Sense" through Mental Multiplication

Update Version: November 6, 2006

An appreciation for mathematics assumes a perception of the properties of numbers and the relationships between them, sometimes called a "number sense," and it is difficult to imagine developing a general skill in mathematics without it. Teachers can encourage this growth through instruction and classroom activities, but acquiring a good number sense is to a large extent an individual process. Manipulating numbers with rote methods of arithmetic (or worse, with an electronic calculator) does little to develop this capability, as these universal methods treat numbers as simply collections of digits.

Mental multiplication is an outstanding activity for developing this number sense. Multiplication has the advantages that it incorporates addition and subtraction, the concept is easily understood and often encountered, and the methods work on both simple and complex numbers. The techniques are just puzzling enough that they often lead to interests in algebra and elementary number theory. Finally, developing this aptitude can be a fun diversion as well as a useful ability for both children and adults. In this spirit, this paper looks at mental multiplication strategies that depend on the properties and relationships of numbers, rather than "tricks" for very specific numbers that are found in many popular books on speed arithmetic.

There are simpler methods here for newcomers to this field, and these can also be taught to children. In addition, there are more complex methods that may be new even to those with a long interest in this area. Following a brief look at the common, straightforward methods of mental multiplication, we will discuss more creative strategies that engage us in the fascinating labyrinth of the number world.

Straightforward Approaches for Mental Multiplication

The written multiplication procedure that is taught in elementary school is a very poor way to perform multiplication mentally, as it involves remembering intermediate digits and produces the result from right to left. The two most common techniques used by "lightning calculators" for mental multiplications are adding "partial products" and performing "cross-multiplication" on the digits.

Partial products are the combinations of the individual digit multiplications, and they are added up from left to right to find the product:

$$\begin{aligned} 46 \times 58 &= 40\text{x}50 + 40\text{x}8 + 6\text{x}50 + 6\text{x}8 \\ &= 2000 + 320 + 300 + 48 \\ &= 2668 \end{aligned}$$

The terms are added as they are calculated, so when 40x8 is calculated, it is added to 2000 to get 2320, then 6x50 is added to get 2620, and finally 6x8 is added to yield 2668. There is only one running total to remember.

Cross-multiplication does not involve these large sums. The digits of the product are found one at a time, but the procedure has the disadvantage that the digits are produced from right to left, so they must be remembered and reversed to recite the answer verbally. In this method the combinations of single-digit products that contribute to each digit of the result are added, including carries. For example,

46 x 58: 6x8 = 48, or **8** with a carry of 4
4x8 + 6x5 + 4 = 66 , or **6** with a carry of 6
4x5 + 6 = **26**
Answer: 2668

Both of these methods have the advantages that they can produce results very quickly with practice, they scale up very well with larger multipliers, and they don't require any multiplications beyond one-digit by one-digit. They are simple and very practical methods.

They do not, however, develop any perception of the properties and relationships of numbers. They are mechanical and, to my mind, a bit dry. The approach promoted here is to creatively restate the problem in another, much easier way that provides greater insight.

General Methods for Multiplying Two-Digit Numbers

There are various ways to simplify multiplication based on the properties and relationships of the numbers involved. Some are easily seen and understood, while others are more surprising and revelatory.

Let's begin with simple insights. We might notice in a problem that one of the multipliers is quite near a very round number, say, a multiple of 10 or 25. We can multiply by that round number instead and adjust for the difference at the end. For example,

29 x 34 = 30x34 − 34

To find 30x34 here, we would multiply from left to right: 30x30 + 30x4. Now if a multiplier exceeds a multiple of 10 by the amount of the multiple, we can use the multiple of 10 and add 1/10 of that result. If a multiplier lies below the multiple of 10, we subtract 1/10 of the result. Multiples of 11 and 9 have these properties.

33 x 62: Find 30x62 = 1860, then 1860 + 186 = 2046
36 x 62: Find 40x62 = 2480, then 2480 − 248 = 2232

We would not subtract 248 directly in the last example, but rather subtract 250 and add 2, a slightly different view of subtraction that makes a large practical difference.

We can also look at a number as a collection of convenient groupings. For example, we can multiply 124726132 by 5 by first halving each even grouping in the first number and then appending zero:

12 4 72 6 132 x 5 = 6 2 36 3 066 0 or 8 32 6 31 x 5 = 4 16 3 15 5

Multiplying a number by 15 can be done by multiplying by 10 and adding half the result. We can think of adding a zero, and then adding half of each even grouping to itself, working left to right and keeping the same number of digits in the grouping as it started with. If a grouping ends up with an additional digit, the upper digit is added to the grouping to the left. The presentation below makes the calculation look more difficult than it actually is—the result is generated smoothly from left to right, with perhaps a correction for a carry from the next grouping, as with the carry of 1 from the (72+36) grouping below:

$$\begin{aligned} 12\ 4\ 72\ 6\ 132 \times 15 &= (12+6)\ (4+2)\ (72+36)\ (6+3)\ (132+66)\ 0 \\ &= \quad 18 \quad\ 7 \quad\ 08 \quad\ 9 \quad\ 198 \quad 0 \end{aligned}$$

Multiplication by 25, or 100/4, can be thought of as appending two zeros and dividing by 4. Multiplying by 50 can be done as 100/2, 75 as 300/4, 125 as 1000/8, and so forth.

These are reasonable and readily understood concepts that involve looking at the whole number rather than individual digits. This is a mental shift that is subtle but critical in developing a number sense. Methods like these are also more general than they seem at first, because if they *almost* apply, we can use them on nearby numbers and then apply a correction at the end.

There are, however, some intriguing methods for multiplying two numbers in certain ranges that encourage an awareness of the interconnectedness of numbers. Consider this collection of unusual algebraic identities, useful for multiplying two numbers near 50 or 100:[1]

$ab = [(a+b)/2 - 25] \mid (50-a)(50-b)$ for **a** and **b** near 50

$ab = (a + b - 100) \mid (100-a)(100-b)$ for **a** and **b** near 100

$ab = (a + b/2 - 50) \mid (50-a)(100-b)$ for **a** near 50 and **b** near 100

In our notation the vertical bar separates two-digit (hundreds) groupings in the result. So the notation "**|n**" represents a two-digit number string. If **n** has more than two digits, they are merged (added) to the digits to the left of the "|" sign. For example, 3|129 = 4|29 = 429. If we end up with a negative value of **n** as in 25|-125, we can borrow 2 from the leftmost group (which is really 200) to make the rightmost one positive, so we convert this to 23|75 = 2375. If we end up with a .5 on the left side, we drop it and add 50 to the right side. In the formulas above the terms can end up negative depending on whether **a** and **b** lie on the low or high side of 50 or 100.

These look tough, but they are not. Here are some examples, sorted by the identity they use:

$43 \times 47 = (45 - 25) \mid 7\times3 = 2021$
$45 \times 53 = (49 - 25) \mid -5\times3 = 24 \mid -15 = 2385$
$43 \times 48 = (45 - 25) \mid (50 + 7\times2) = 2064$

$93 \times 97 = 90 \mid 7\times3 = 9021$
$108 \times 94 = 102 \mid -8\times6 = 10152$

$43 \times 96 = (43 + 48 - 50) \mid 7\times4 = 4128$

Clearly we are now glimpsing the ghost in the machine. What's behind these odd formulas? We can show that they are true algebraically, but we would never come across such things using rote methods for arithmetic. Conceptual breakthroughs occur when attempts are made to shorten or

[1] I have only seen this presented by R. Gibert on the alt.math.recreational newsgroup.

circumvent standard procedures for some reason, and mental math is an activity of this kind that is familiar and interesting to even the youngest student.

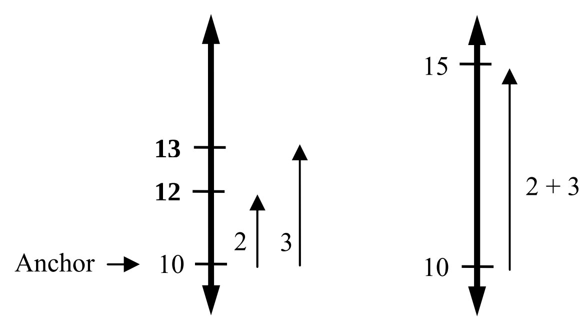
These methods all involve thinking about the properties of numbers, so they appeal to me as methods for somewhat specific circumstances. However, there is a type of method that is useful in a very wide variety of multiplications. When the multipliers are a distance **c** and **d** from a round number, their product can be represented by the product of the round number and the sum of the round number and the two differences, with the product of the two differences added at the end as a small correction. There does not seem to be a consistent name for this method in the literature; I call it the "Anchor Method":

$$(a+c)(a+d) \;=\; a(a+c+d) + cd \qquad \textbf{\textit{Anchor Method}}$$

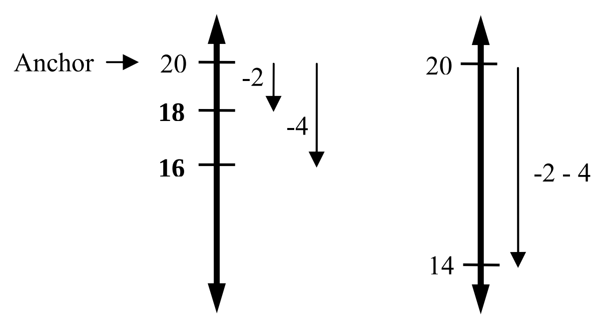
This is much easier to use than it might appear, and a knack for it is easily developed with a small amount of practice. The concept can be taught to children. I visualize "anchoring" one multiplier at the round number, and then literally stringing out the differences from the original numbers from this anchor to find the other multiplier. It will turn out that the original multipliers move outward, their product will be less than the original, so the correction at the end needs to be added, and if they move inward, the correction is subtracted. This corresponds to the intuitive (and correct) concept that a square has the greatest area for a given sum of side lengths; the rectangle produced by shifting length from one side to another side will have a smaller product of the two sides because $(x+n)(x-n) = x^2 - n^2$ is always less than x^2.

Below are three representative problems and a visualization of each solution.[2]

$12 \times 13 \;=\; 10 \times 15 \;+\; 2 \times 3 \;=\; 156$

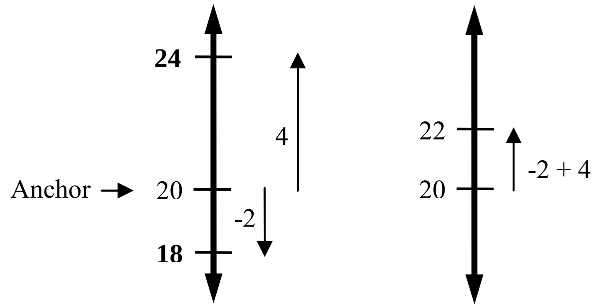


$18 \times 16 \;=\; 20 \times 14 \;+\; 2 \times 4 \;=\; 288$



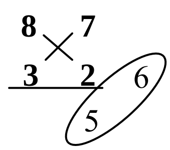
[2] The numbers are shown on vertical number lines because I "see" number lines as vertical rather than horizontal. I remember having difficulty learning the number line concept in grade school, and I believe it was due to the horizontal layout in textbooks. I understand the formatting difficulties, but a vertical layout would be much more intuitive to children (and me) who think numbers go up as they get higher.

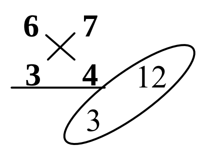
18 x 24 = 20 x 22 − 2 x 4 = 432



Notice that we have now progressed from looking at a number as a whole to visualizing distances and relationships *between* numbers. This is another critical step in developing a number sense.[3]

Children today find it fascinating that schoolchildren were not always taught their multiplication tables past 5x5. In the 1800's in America, they were often taught to use the scheme below for the higher multiples. In the left diagram, 3 and 2 are (10-7) and (10-8), 5 is either (7-2) or (8-3), and 6 is (3x2), giving 8x7=56. Here 10 as used as an anchor, since 8x7 = 10x(10-2-3) + 2x3. If the product of the differences is a two-digit number, the higher digit is added to the other digit, as in the answer 42 for the diagram on the right:





This is an excellent way to generate interest in the Anchor Method for the classroom. I have often wondered whether children should be taught their multiples of 12 this way rather than by rote: 12x12 = 10x14 + 2x2, 8x12 = 10x10 − 2x2, 9x12 = 10x11 − 2x1 = 108, and so forth.

An anchor of 100 is very common, say, $84^2 = 100\text{x}68 + 16^2$. With 100 as the anchor, we can find 68 as the last digits of 84 doubled rather than by finding the difference between 100 and 84 and subtracting this from 84.

If the numbers to multiply are far apart, though, we can end up with a large correction term **cd**. There are a few strategies to deal with this:

1. Subtract one number from a very round number (or add it to a very round number) to bring it closer to the other number:

   23 x 67 = 23(100−33) = 2300 − 23x33 = 2300 − (20x36 + 3x13)

[3] A visual approach to mental calculation has not been the norm historically among "lightning calculators," who typically were auditory in nature, demonstrated by their muttering, standing, sitting, and gesticulating while solving a problem. It is startling, actually, to notice the same physical reactions commonly in children doing mental arithmetic (but not typically permitted in the classroom!). However, those who were mathematicians (including Wallis, Gauss, Ampere, von Neumann, and Aitken), each with a highly developed number sense, could produce prodigious results without exhibiting these physical effects, and would seem to have developed a visual perception.

2. Divide or multiply one number by a low integer and add a correction:

$$23 \times 67 = 23x33x2 + 23 = 2(20x36 + 3x13) + 23$$

3. Break one number into two convenient parts:

$$23 \times 67 = 23(50+17) = 2300/2 + 23x17 = 1150 + 20^2 - 3^2$$

In the end we can use our creativity and experience to manipulate the calculation as we wish.[4]

So we have progressed from a fully mechanical view of multiplication to one in which the inherent properties of a number are recognized and utilized, and finally to a more nuanced approach that takes advantages of the relationships between numbers as well. Experience in these types of calculation rapidly leads to an appreciation of the character of the number spectrum, and we eventually develop individual preferences for solving particular types of calculations. Often experienced calculators opt for multiplication methods that involve squaring numbers, because there are a number of techniques for simplifying the squaring process. This topic is taken up in the following sections.

The Importance of Squares

Squares often appear in solutions to physical or mathematical problems, so opportunities to mentally calculate them spontaneously occur. They also arise in intermediate steps of more complicated calculations, such as square root and cube root algorithms, power series formulas for approximating functions such as logarithms and exponentials, and various approximation formulas for specific functions.

However, squares appear most often in mental calculations when performing straightforward multiplications. Here we transform the multiplication into a number squared plus a simple arithmetic adjustment, and this opens another world of possibilities for creative and very useful strategies for multiplications.

One of the most powerful tools in mental calculation is converting the multiplication of two different numbers into the square of the average minus the square of the distance to the average. This is shown by the "Midpoint Method," an algebraic identity:

$$(a+c)(a-c) = a^2 - c^2 \qquad \textbf{\textit{Midpoint Method}}$$

where **a** is the average of the two numbers, **(a+c)** is one of the numbers, and **(a−c)** is the other number. This is algebraically equivalent to the Anchor Method formula if **d** = **-c**, or in other words when the anchor is midway between the two multipliers. The choice of the anchor as the midpoint or some other number depends on the problem and on personal preferences, but there is no doubt that using the midpoint is a very common technique. For example,

$28x32 = 30^2 - 2^2$
$52x78 = 65^2 - 13^2$

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[4] John McIntosh (www.urticator.net) points out that experienced calculators have a familiarity with convenient multiples, and 67x3 = 201 can be used here: 23x67 = (23x201)/3 = 4623/3 = 1541.

or, considering the problem in the first section of this paper,

$$46 \times 58 = 52^2 - 6^2$$

Less convenient multipliers can be manipulated in a number of ways to use this technique. We might have the case where there is no midpoint of the two multipliers—here we can adjust one of the multipliers by one, do the calculation, and then provide a correction to account for the original adjustment, as for $28 \times 33 = 28 \times 32 + 28 = 30^2 - 2^2 + 28$, but in this particular case it may be easier to use the Anchor Method from the last section: $28 \times 33 = 30 \times 31 - 2 \times 3$.

The Squares of Two-Digit Numbers

Let's consider the squares of two-digit numbers first. If we use the Midpoint Method to convert two-digit multiplications to a difference of squares, we can reduce all the possible combinations of two-digit by two-digit multiplications to just the two-digit squares plus minor arithmetic adjustments—10,000 different multiplications to just 100!

But we have to calculate the squares, of course. Fortunately, there are excellent methods to simplify squaring. One strategy is to use the Midpoint Method *in reverse*. We can split a square into the product of two numbers equidistant from the original number, and add the square of that distance, again one scenario of the Anchor Method. For example, let's continue with one of our examples from earlier:

$$52 \times 78 = 65^2 - 13^2$$

Now we find 65^2 by spreading 65 in both directions by an equal amount and adding the square of that amount. Here a good spread is by 5, yielding $65^2 = 60 \times 70 + 25 = 4225$. Similarly, $13^2 = 10 \times 16 + 9 = 169$. So we can turn a general multiplication into a square plus a small correction, and we can turn that square into an even simpler multiplication and one more small correction if needed. Again, I find it helpful to remember that the average squared will always be larger than the spread numbers multiplied, so when collapsing two multipliers to a square you *subtract* the correction, and when spreading a square to the product of two numbers you *add* the correction. These transformations become automatic and very fast after a bit of practice. You will also develop a number sense, a real feeling of the relationships between numbers, and you'll naturally come to learn a good number of two-digit squares by heart, which will only increase the speed of these calculations.

Those with an interest in mental calculation may recognize in the example of 65^2 the trick for squaring numbers ending in 5: multiply the number left of the units digit by that number plus one, and then append 25, as in 6x7 | 25 = 4225. Now we can see why that works.

The Midpoint Method described earlier applies to larger numbers, e.g., $244 \times 376 = 310^2 - 66^2$. But 310^2 is really just a square of a two-digit number followed by two zeros—what if we had ended up with a three-digit square here? Again we split the square into two numbers equidistant from the original number, adding the square of that distance. To illustrate, $244 \times 382 = 313^2 - 69^2 = [300 \times 326 + 13^2] - 69^2$, and we end up with a simple calculation if we know the two-digit squares.

And there are indeed a variety of other techniques for finding squares. Most of these involve expressing the number to be squared as the sum of two other numbers that are more easily squared, using the binomial expansion for squares:

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \textbf{\textit{Binomial Expansion for Squares}}$$

To illustrate, $34^2 = (30+4)^2 = 30^2 + 2x30x4 + 4^2 = 1156$, and $69^2 = (70-1)^2 = 70^2 - 2x70x1 + 1^2 = 4761$, and $313^2 = (300+13)^2 = 300^2 + 2x300x13 + 13^2 = 90000 + 7800 + 169 = 97969$.

In another application of the binomial expansion, one of the most intriguing and useful techniques easily finds the square of a number near 50. Here we add the difference from 50 to 25, multiply by 100, and add the difference squared. If the number is within 10 of 50, we can add the difference to 25 and simply append the distance squared rather than adding it. In our notation,

$$(50+a)^2 = (25 + a) \mid a^2 \qquad \text{Ex: } 52^2 = (25+2) \mid 2^2 = 2704$$
$$44^2 = (25-6) \mid 6^2 = 1936$$

This is a simpler way of thinking of the binomial expansion $(50+a)^2 = 2500 + 100a + a^2$.

We can also use the fact that multiples of 25 are fairly round numbers. We can square numbers near 25 using the expansion $(25+a)^2 = 625 + 50a + a^2$, as $27^2 = 625 + 100 + 4 = 729$. The relation $(75+a)^2 = 5625 + 150a + a^2$ can be used to find, say, $78^2 = 5625 + 450 + 9 = 6084$. We can re-format these into our notation, remembering that a .5 in a group is converted to a 50 in the group to the right of it:

$$(25+a)^2 = (6 + a/2) \mid (25 + a^2) \qquad \textit{Ex: } 27^2 = (6+1) \mid (25 + 2^2) = 729$$
$$(75+a)^2 = (56 + a + a/2) \mid (25 + a^2) \qquad \textit{Ex: } 78^2 = (56+3+1.5) \mid (25 + 3^2) =$$
$$60.5 \mid 34 = 6084$$

Alternatively, we can re-arrange the binomial expansion of two-digit squares ending in 9, 8, or 7 in another interesting way:[5]

$$(10a+9)^2 = 100a(a+1) + \mathbf{8}0(a+1) + \mathbf{1}$$
$$(10a+8)^2 = 100a(a+1) + \mathbf{6}0(a+1) + \mathbf{4}$$
$$(10a+7)^2 = 100a(a+1) + \mathbf{4}0(a+1) + \mathbf{9}$$

where the digits in bold comprise the square of the units digit. So $79^2 = 5600 + 640 + 1 = 6241$, $87^2 = 7200 + 360 + 9 = 7569$, and so on.

If a neighbor of the number has a square that is known or easily calculated, we can use this convenient square and adjust for the difference. Since $(a+1)^2 = a^2 + a + (a+1)$, we can find $31^2 = 30^2 + 30 + 31 = 961$. Similarly, $29^2 = 30^2 - 30 - 29 = 841$. If we let "‖n" represent a single-digit number string, we can use this notation for a number ending in 1: $(n1)^2 = n^2 \parallel 2n \parallel 1$, so $31^2 = 9 \parallel 6 \parallel 1 = 961$ and $71^2 = 49 \parallel 14 \parallel 1 = 5041$. We can represent the square of a number ending in 9 as $(n9)^2 = (n+1)^2 \parallel -2(n+1) \parallel 1$, so $69^2 = 49 \parallel \mathbf{-}14 \parallel 1 = 4761$. For other neighboring numbers we can find the square of the convenient number, then add or subtract the original number, the final number, and twice each number in between, so $32^2 = 30^2 + 30 + 2x31 + 32 = 1024$, a square that we recognize from powers of 2.

[5] I have only seen this presented by "jpam89" on the Yahoo Mental Calculation group.

We can also use techniques that "throw" the number to a different number that is easier to square. As we did before, we can subtract a number from a very round number and use the standard binomial expansion $(a+b)^2 = a^2 + 2ab + b^2$, so $88^2 = (100-12)^2 = 10000 - 2(1200) + 12^2 = 7744$. Perhaps we become familiar with the squares of numbers less than 25; then a number greater than 25 can be thrown to the low side of 25 using the expansion:

$(25+a)^2 = (25-a)^2 + 100a$ *Ex:* $33^2 = 17^2 + 800 = 289 + 800 = 1089$
$(50+a)^2 = (50-a)^2 + 200a$ *Ex:* $82^2 = 18^2 + 200x32 = 324 + 6400 = 6724$

But wait—we see that 82 is 32 more than 50, and we just learned $32^2 = 1024$, so we can use the relation for squaring numbers near 50 to find $82^2 = (25+32) \mid 32^2 = 57 \mid 1024 = 6724$. Or $82^2 = 80^2 + 2x82 + 2^2$, or $82^2 = 80x84 + 2^2$, or $82^2 = 100x64 + 18^2$. Different approaches spring up as experience grows. Ultimately we will find that the field is quite crowded for squaring numbers less than 100, and in a surprising development we eventually start looking to three-digit numbers for more interesting challenges.

Squares and Products of Three-Digit Numbers

All of the squaring methods given for squaring two-digit numbers have analogies for squaring three-digit numbers. Squares of larger numbers can also use the binomial expansion $(a+b)^2 = a^2 + 2ab + b^2$, where the **2ab** multiplication can again be simplified using squares. If the number has three digits, the middle digit can be included in either **a** or **b**, depending on which squares are most familiar to the mental calculator. Let's write the binomial expansion in our notation so that we don't have to worry about the powers of 10:

$(100a+b)^2 = a^2 \mid 2ab \mid b^2$ for a two-digit b
$(10a+b)^2 = a^2 \mid (20ab + b^2)$ for a one-digit b

To find 756^2, we can convert this to $(700+56)^2 = 7^2 \mid 2x7x56 \mid 56^2$. This is convenient because we know how to square numbers near 50 from the last section: $56^2 = (25+6) \mid 6^2 = 3136$. So we have $49 \mid 784 \mid 3136 = 56 \mid 115 \mid 36 = 571536$. But finding 75^2 is even easier since it ends in 5, so we might choose to let $756^2 = (750+6)^2 = 75^2 \mid (20x75x6 + 6^2) = 5625 \mid 9036 = 56 \mid 115 \mid 36 = 571536$.[6]

Now three-digit numbers can be treated like two-digit numbers in all the methods in the previous section if we treat the leftmost two digits as a single digit, as in using the technique for squaring numbers ending in 5 to find $235^2 = 23x24 \mid 25 = 55225$. We can also alter some of the methods slightly for three-digit calculations. The square of a number near 500 can be found by adding the difference from 500 to 250 and appending the difference squared as a three-digit value, where here we introduce the comma "," as the separator for 3-digit groupings:

$$(500+a)^2 = (250 + a)\ ,\ a^2$$

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[6] If squares of three-digit number arise very often, it might prove useful for the more experienced calculator to rearrange the binomial expansion for some convenient numbers. For example,

$(n25)^2 = (n^2 + n/2) \mid 06 \mid 25$ *Ex:* $325^2 = (9 + 1.5) \mid 06 \mid 25 = 105625$
$(n75)^2 = (n^2 + n + n/2) \mid 56 \mid 25$ *Ex:* $675^2 = (36 + 6 + 3) \mid 56 \mid 25 = 455625$

and more generally, for positive or negative **a**,

$(n25 + a)^2 = (n^2 + n/2) \mid (2an + a/2) \mid (625 + a^2)$ *Ex:* $328^2 = 10.5 \mid 19.5 \mid 634 = 107584$
$(n75 + a)^2 = (n^2 + n + n/2) \mid (2an + a + a/2) \mid (5625 + a^2)$ *Ex:* $683^2 = 45 \mid 108 \mid 5689 = 466489$

so,

$$513^2 = 263{,}169$$
$$492^2 = 242{,}064$$

Numbers greater than 250 can be thrown to the low side of 250, and the same for 500, by modifying our earlier techniques for 25 and 50:

$$(250+a)^2 = (250-a)^2 + 1000a$$
$$(500+a)^2 = (500-a)^2 + 2000a$$

The product of two different three-digit numbers can be reduced by the Midpoint Method to the difference of two three-digit squares at most. This is illustrated by the particularly difficult example given below. Here we are using the relation $(100a+b)^2 = a^2 \mid 2ab \mid b^2$. Notice that rather than calculating the two squares separately and subtracting them, we subtract their individual hundreds groups before we merge them:

$$\begin{aligned} 327 \text{ x } 755 &= 541^2 - 214^2 \\ &= (500+41)^2 - (200+14)^2 \\ &= (5^2 - 2^2) \mid (10\text{x}41 - 4\text{x}14) \mid (41^2 - 14^2) \\ &= 21 \mid 354 \mid 1485 \\ &= 246885 \end{aligned}$$

But there are other ways to do this particular problem. Let's see, 327 = (300 + 30 − 3), so we can find y = 300x755 and then add y/10 and subtract y/100. Or we could double 327 and use the Anchor Method to find 654x755 = 700x709 − 46x55 and then halve the result. Or maybe you have a better idea.

Squares and Products of Four-Digit Numbers

The square of a four-digit number can be readily simplified using the Anchor Method if the number lies near a multiple of 1000, as $6888^2 = 7000\text{x}6776 + 112^2$ = (7x6000 + 7x700 + 7x70 + 7x6),000 + $(100\text{x}124 + 12^2)$ = 47432,000 + 12544 = 47444544. Now let **a** denote the first two-digit half of the number, and **b** the second two-digit half of it. If **a** or **b** is small, we might use the binomial expansion $(a \mid b)^2 = a^2 \mid 2ab \mid b^2$, so $6808^2 = 68^2 \mid 2\text{x}68\text{x}8 \mid 8^2$ = 4624 | 1088 | 64 = 46348864. For the more difficult problem of multiplying two different four-digit numbers, there is a new method (more mechanical in nature than the others, unfortunately) that we will derive here. We will find that this new method is also useful for squaring a four-digit number when the difference between **a** and **b** is small.

So let's consider the general problem of four-digit multiplication. If we have developed the ability to multiply two-digit numbers, we could use partial products to perform the multiplication. If **a** and **b** are two-digit halves of a four-digit number, and **c** and **d** are two-digit halves of a four-digit number, we can express the four partial products ac, ad, bc, and bd, as:

$$a|b \text{ x } c|d = ac \mid (ad + bc) \mid bd$$

This requires four two-digit multiplications. Surprisingly, there is a different expansion that requires only three two-digit multiplications: ac, bd, and (a-b)(c-d).

$$a|b \text{ x } c|d = ac \mid (ac + bd - (a-b)(c-d)) \mid bd$$

Of course, there are a couple of subtractions involved. However, this method really shines when (a-b) or (c-d) is small.

This equation can also be manipulated to simplify even further the multiplication of four-digit numbers:[7]

$$a|b \text{ x } c|d = 101 \text{ x } ac|bd - 100 \text{ x } (a-b)(c-d)$$

This arrangement provides a convenient method for multiplying two four-digit numbers. Here are the steps, using an example of 6143x2839, or 61|43 x 28|39 in place of a|b x c|d. We assume that the two numbers are in view so we don't have to memorize them:

<u>6143 x 2839:</u>

1. Find ac = 61x28 = 1708. Find bd = 43x39 = 1677.

2. Merge them as ac | bd: 1708 | 1677 = 17 | 24 | 77

At this point, all we have to remember is this set of three numbers. There is no need to remember anything from step 1.

3. Now let's take the sum of the last two numbers, 24+77, and subtract (a-b)(c-d). Here we see that a>b but c<d, so we can just take the positive differences and change the subtraction to an addition:

$$24 + 77 + (61-43)(39-28) = 101 + 18\text{x}11 = 299$$

We need to remember this number, too.

4. Now the answer is:

$$17 \mid (17 + 24) \mid 299 \mid 77$$

where the first number is the leftmost number of step 2, the second number is the sum of the leftmost and middle number of step 2, the third number is the one we calculated in step 3, and the fourth number is the rightmost number in step 2. Merging this in our head, we give the result as 17439977.

When multiplying four-digit numbers, there may be some shortcut available depending on the particular numbers in the problem, but this is a good general method.

This method can also be used to square a four-digit number; here **a** = **c** and **b** = **d**, and we have:

$$(a|b)^2 = 101 \text{ x } a^2|b^2 - 100 \text{ x } (a-b)^2$$

[7] This method is discussed in more detail in another paper of mine, *A Method for 4x4-Digit Mental Multiplications*, located on the http://www.myreckonings.com site.

Now for squares there are only three partial products anyway in its binomial expansion, as given by $(a \mid b)^2 = a^2 \mid 2ab \mid b^2$. However, if **a** is approximately equal to **b**, there is a real advantage to using this new method, as the last term $(a-b)^2$ will be quite small. Say we are finding 6368^2:

$\underline{6368^2}$:

1. $a^2 \mid b^2 = 63^2 \mid 68^2 = 3969 \mid 4624 = 40 \mid 15 \mid 24$.

   It is easiest to calculate sums from left-to-right: $69 + 46 = (60 + 40) + 9 + 6 = 115$.

2. Find the sum of the last two numbers minus $(a-b)^2 = 15 + 24 - 5^2 = 14$.

3. The answer is $40 \mid (40 + 15) \mid 14 \mid 24 = 40551424$, where the first group is the same as in step 1, the second group is the sum of the first two groups in step 1, the third group is the number found in step 2, and the last group is the same as in step 1.

Perhaps it would be a good idea to conclude by reviewing the methods for two-digit squares that might be used here. I found 63^2 by the method for squaring numbers near 50, so $63^2 = (25+13) \mid 169 = 3969$. There were a few possibilities for finding 68^2:

a) I could use the method for squaring numbers near 50, giving $68^2 = (25+18) \mid 18^2 = 4624$ after I spread out $18^2 = 20\text{x}16 + 2^2 = 324$.
b) I remember that $32^2 = 1024$, so I could throw 68 to the other side of 50, yielding $68^2 = (50-18)^2 + 200\text{x}18 = 1024 + 3600 = 4624$, or I could spread 68 to an anchor of 100 to get $68^2 = 100\text{x}36 + 32^2$.
c) I might use the binomial expansion $60^2 + 2\text{x}8\text{x}60 + 8^2 = 3600 + 960 + 64 = 4624$. Even easier, I might use the expansion $(10a+8)^2 = 100a(a+1) + \mathbf{6}0(a+1) + \mathbf{4}$, which I remember because $\mathbf{64} = 8^2$, to get $4200 + 420 + 4 = 4624$. Or I could expand $(70-2)^2 = 70^2 - 2\text{x}70\text{x}2 + 2^2 = 4900 - 280 + 4 = 4624$, or $(65+3)^2 = 65^2 + 2\text{x}65\text{x}3 + 3^2 = 4225 + 390 + 9 = 4624$, and so on.
d) I could use the relation for squaring numbers near 75, giving $68^2 = (75-7)^2 = (56-7-7/2) \mid (25+7^2) = 45.5 \mid 74 = 45 \mid 124 = 4624$.

The last one is not the most efficient, but mental calculators will tell you that once you start a calculation, you don't change horses in midstream. Experience and personal preferences dictate our choice, as we have an abundance of strategies now for squaring numbers.

So we have a new method described above for multiplying two four-digit numbers. To square a four-digit number, we first look at the number that needs to be squared. If the number lies relatively close to a multiple of 1000, we can use the Anchor Method to simplify the problem. If either of the two-digit halves, **a** or **b,** is small, say 07 or 12, it is probably most useful to use the expansion $(a \mid b)^2 = a^2 \mid 2ab \mid b^2$. If the difference between **a** and **b** is small, however, the new method given above can provide a real advantage.

Mental Multiplication and the "Number Sense"

Mental multiplication is a highly creative and satisfying endeavor, offering a variety of interesting strategies, more than I have presented here and many more than most people realize. It is a skill that engages both children and adults, and one that naturally leads to a real familiarity with the properties and relationships of numbers. It provides a useful and fun approach for developing a number sense and generating a true appreciation for the elegance of elementary mathematics.