

# Mentally Expressing a Number as a Sum of Four Squares

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The fact that every positive integer can be written as the sum of four squares is called Bachet's Theorem, since he first stated it explicitly in 1621. Lagrange supplied the first proof in 1770. This paper provides some recommendations on mentally solving this problem.

Reducing a number to one or more sums of four squares is historically a common problem posed to lightning calculators [1]. For example, it was reported in 1911 that the calculator Gottfried Ruckle was asked to reduce  $N = 15663$  as a sum of four squares. He produced a solution of  $125^2 + 6^2 + 1^2 + 1^2$  in 8 seconds, followed immediately by  $125^2 + 5^2 + 3^2 + 2^2$ . A more difficult problem (reflected by a first term that is farther from the original number, with correspondingly larger later terms) took 56 seconds:  $11399 = 105^2 + 15^2 + 8^2 + 5^2$ . In general, the strategy is to begin by setting the first term to be the largest square below  $N$  and try to represent the smaller remainder as a sum of three squares. Then the first term is set to the next largest square below  $N$ , and so forth. Over time a lightning calculator would become familiar with expressing small numbers as sums of squares, which would speed up the process.

In a posting on the Yahoo Mental Calculation group, Oleg Stepanov [2] provided examples of cases where breaking  $N$  into the sum of four squares resulted in the largest coefficient that is far from the square root of  $N$ . For example, the square root of 1536 is just over 39, yet  $1536 = 32^2 + 16^2 + 16^2 + 0^2$ . Presumably, these would be the most difficult to find solutions for, since one usually starts testing with the integers around the square root to eliminate the possible ranges of the other coefficients. Stepanov provided a list of every 4-digit integer  $N$  for which the largest coefficient lies at least 6 from its square root (there are 19), with 1536 being the first one in the table.

<b>N</b>	<b>Integer Nearest <math>N^{1/2}</math></b>	<b>Sum of 4 Squares with Largest Coefficient Furthest from <math>N^{1/2}</math></b>
1536	39	$32^2 + 16^2 + 16^2 + 0^2$
2176	46	$40^2 + 24^2 + 0^2 + 0^2$
2944	54	$48^2 + 24^2 + 8^2 + 0^2$
3072	55	$48^2 + 16^2 + 16^2 + 16^2$
3584	59	$48^2 + 32^2 + 16^2 + 0^2$
3968	62	$56^2 + 24^2 + 16^2 + 0^2$
4544	67	$60^2 + 28^2 + 12^2 + 4^2$
4992	70	$64^2 + 24^2 + 16^2 + 8^2$
5120	71	$64^2 + 32^2 + 0^2 + 0^2$
5632	75	$64^2 + 32^2 + 16^2 + 16^2$
6144	78	$64^2 + 32^2 + 32^2 + 0^2$

6848	82	$76^2 + 28^2 + 12^2 + 12^2$
7424	86	$80^2 + 32^2 + 0^2 + 0^2$
7552	86	$80^2 + 32^2 + 8^2 + 8^2$
7680	87	$80^2 + 32^2 + 16^2 + 0^2$
8704	93	$80^2 + 48^2 + 0^2 + 0^2$
8960	94	$88^2 + 24^2 + 24^2 + 8^2$
9088	95	$88^2 + 32^2 + 16^2 + 8^2$
9664	98	$92^2 + 28^2 + 20^2 + 4^2$

For mental calculation, it might seem practical to memorize these 19 cases. However, we can notice (by checking the last two digits only) that all of the numbers in the table above are divisible by 4. It is therefore possible to divide N by 4 and then find the sum of squares for this N/4 to make the process easier [3]:

$$\text{If } N/4 = a^2 + b^2 + c^2 + d^2 \text{ then } N = (2a)^2 + (2b)^2 + (2c)^2 + (2d)^2$$

so taking N=1536, we get N/4=384, which is easier to break into squares:

$$N/4 = 16^2 + 8^2 + 8^2 + 0^2,$$

and now the highest square is only 3 away from the square root. We then multiply each coefficient by 2 to get

$$N = 32^2 + 16^2 + 16^2 + 0^2$$

which matches the table above.

Actually,  $384/4 = 96$ , and  $96/4 = 24$ , and  $24/4 = 6$ , so we can reduce N to  $N/4^4 = 6$ . Then we easily find that

$$6 = 2^2 + 1^2 + 1^2 + 0^2$$

and we can multiply each coefficient by  $2^4$  to get the result for N. In fact, all of the numbers in the table are divisible by at least  $4^3$ , so this seems related to the first coefficient being so far from the square root. This means that there is no reason to memorize these situations, but rather divide out the 4's before starting, and then fix the answer at the end. (It also means that we have discovered probably the most complicated test in the world for checking divisibility by 4.)

If N is divisible by 2 but not by 4, then we can use N/2 to simplify the process [3]:

$$\text{If } N/2 = a^2 + b^2 + c^2 + d^2 \text{ then } N = (a+b)^2 + (a-b)^2 + (c+d)^2 + (c-d)^2$$

It is also true that generally there is more than one way to break a number into the sum of four squares. So it would be nice to know if there are additional ways after the first one is found. We can use the Jacobi Four Square Theorem for this:

*The number of representations of an integer as the sum of four squares is equal to eight times the sum of all its divisors that are not divisible by 4.*

The representations are the many different ways the sums can be arranged. For example, the expression  $1^2 + 2^2 + 3^2 + 4^2$  can have these squares in different orders ( $4! = 4 \times 3 \times 2 \times 1 = 24$  representations) and it can also have  $(-1)^2$  instead of  $1^2$ , and so forth, which gives  $2^4 = 16$  combinations, so a solution with 4 unique squares has  $24 \times 16 = 384$  representations.

Of course, if a square is duplicated, or if one of the squares is 0 (which has no negative), then there are not as many representations possible. The 11 different possible types and the number of their representations are:

Type	Expression	Representations (R)
1	$a^2 + b^2 + c^2 + d^2$	384
2	$0^2 + a^2 + b^2 + c^2$	192
3	$a^2 + a^2 + b^2 + c^2$	192
4	$a^2 + a^2 + b^2 + b^2$	96
5	$0^2 + a^2 + a^2 + b^2$	96
6	$a^2 + a^2 + a^2 + b^2$	64
7	$0^2 + 0^2 + a^2 + b^2$	48
8	$0^2 + a^2 + a^2 + a^2$	32
9	$0^2 + 0^2 + a^2 + a^2$	24
10	$a^2 + a^2 + a^2 + a^2$	16
11	$0^2 + 0^2 + 0^2 + a^2$	8

Other than memorizing it, the way to remember these values is the same way they are derived:

- Let: A = number of "a" terms
- B = number of "b" terms
- C = number of "c" terms
- D = number of "d" terms
- E = number of "0" terms
- n = the number of non-zero terms

Then the number of representations is given by

$$R = 2^n \times 24 / (A! \times B! \times C! \times D! \times E!)$$

where the factorial (!) means to multiply by all the integers between that number and 1, so for example  $3! = 3 \times 2 \times 1$ . If there are no terms of a given letter, we just ignore it, since  $0! = 1$ .

For example, if we find a sum of four squares such as we have now:

$$1536 = 32^2 + 16^2 + 16^2 + 0^2$$

then the number of representations from this solution is

$$R = 2^3 \times 24 / (1! \times 2! \times 1!) = 8 \times 24 / 2 = 96$$

which correctly corresponds to Type 5 ( $0^2 + a^2 + a^2 + b^2$ ).

Let's factor N and use the theorem to find the total number of representations for all the solutions. This gives  $1536 = 2^9 \times 3$ . The divisors of 1536 are the combinations of these, along with 1 and 1536. All combinations that involve powers of 2 greater than one will be divisible by 4, so we can eliminate those. Also, 1536 is divisible by 4. So we are left with the divisors 1, 2, 3, and 6. Then the theorem says that  $R = 8 \times (1+2+3+6) = 96$ .

Our first solution was  $1536 = 32^2 + 16^2 + 16^2 + 0^2$ , which as we saw above allows 96 representations. So there are no more ways that 1536 can be expressed as a sum of four squares, and we can stop looking because we are finished.

However, many N's will have more than one solution. We can look at  $N = 9088$  from the first table.  $N/4^3 = 142$ , so we find the sum of squares for 142:

$$N/4^3 = 11^2 + 4^2 + 2^2 + 1^2$$

Next we multiply each coefficient by  $2^3$  to get our first reduction of N to the sum of four squares:

$$9088 = 88^2 + 32^2 + 16^2 + 8^2$$

Now  $9088 = 2^7 \times 71$ . Since powers of 2 greater than one are divisible by 4, and 9088 is divisible by 4, then the divisors that are not divisible by 4 are 1, 2, 71, and 142. The theorem says that the number of representations from all the possible sum-of-four-squares solutions is

$$R = 8 \times (1 + 2 + 71 + 142) = 1728$$

The first solution we found above is Type 1, so it uses up 384 representations. We are therefore left with 1344 more that are possible. This must be, say, at least four more Type 1 solutions plus a Type 2 or Type 3, or perhaps other types. So there are at least five more possible solutions, and even more if they are of those other types!

Actually, there are tests to see if solutions exist that are the sum of two squares (Types 7 or 9) or that are the sum of three squares (Types 2, 5, or 8) for any  $N$ . These are helpful as well in finding sums after a trial first term is subtracted from  $N$ .

*If  $N$  is of the form  $4^p \times (8k+7)$ , then it cannot be expressed as the sum of three squares--otherwise it can.*

Our  $N = 9088 = 4^3 \times 142$ , and  $142 = 8(17) + 6$ , so it can be expressed as the sum of three squares. Trying some more numbers, we can find a three-square solution as

$$9088 = 72^2 + 48^2 + 40^2 + 0^2$$

To tell if  $N$  can be expressed as the sum of two squares, factor  $N$  into powers of primes. Ignore any powers of 2.

*If every prime factor of  $N$  of the form  $(4k+3)$  occurs an even number of times, then  $N$  can be written as the sum of two squares-- otherwise it cannot.*

Our  $N = 2^7 \times 71$ . Neglecting the powers of 2, we are left with  $71 = 4(17) + 3$ , so it is the type of prime we are interested in, but it has an odd power of 1 in the factorization, so 9088 cannot be expressed as the sum of two squares.

There is no convenient algorithm (beyond the simple one mentioned in the second paragraph of this paper) for finding additional solutions that are indicated by the calculation of representations, but perhaps this will streamline the search by giving an idea of what kinds of solutions do and do not exist.

There are a few other interesting facts related to this topic. There exist other theorems that state that every integer can be written as a sum of four particular multiples of squares. For example, every integer can be written as  $N = a^2 + 2b^2 + 4c^2 + 14d^2$ . There are 54 cases like this that are true for all integers, and Ramanujan provided the complete list in the year 1917.

### References:

- [1] Smith, Steven B. *The Great Mental Calculators*, Columbia University Press, New York, 1983, pp. 158-162.
- [2] <http://users.lk.net/~stepanov/>
- [3] <http://mathforum.org/library/drmath/view/51591.html>