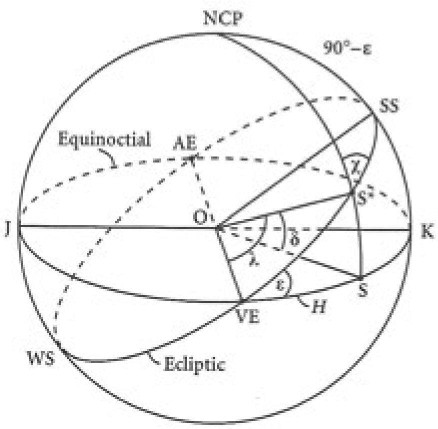
The Analemmas of Vitruvius and Ptolemy

Dead Reckonings: Lost Art in the Mathematical Sciences
http://www.myreckonings.com/wordpress
November 29, 2007



Have you ever had to calculate the positions of astronomical objects? Orbital calculations relative to an observer on the Earth require derivations and time-consuming solutions of spherical trigonometric equations. And yet, these kinds of calculations were accomplished in the days prior to the advent of calculators or computers!

For example, to find the zenith angle (angle to overhead) and azimuth (angle from North) of the sun at any day and time of the year for any location on Earth, the laws of spherical trigonometry produce the formulas below. Here the solar declination δ is a function of the solar longitude λ and ecliptic angle ε as shown in the figure to the left.

$$z_H = \arccos\left(\sin\phi \sin\delta + \cos\phi \cos\delta \cos\tau\right)$$

$$A_H = \arccos\left(\frac{\sin\delta\cos\phi - \sin\phi\cos\delta\cos\tau}{\sin z_H}\right)$$

where: ϕ = terrestial latitude

δ = current solar declination

$\tau = (n/6)\arccos\left(-\tan\phi\tan\delta\right)$

n = the number of unequal hours before or after local noon

z_H = the zenith angle of the sun

A_H = the altitude of the sun above the horizon

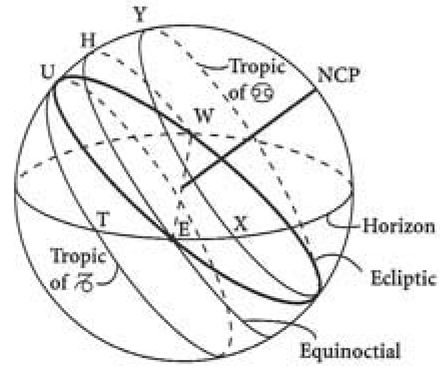
These calculations can be automated today—but did I mention that these solutions were found before electronic calculators?

... or slide rules, or logarithms?

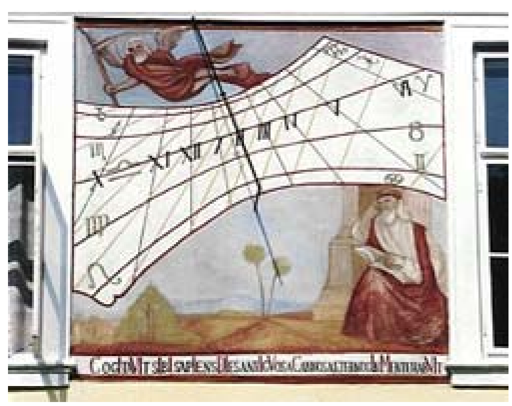
... or trigonometric formulas?

... or even *algebra*??

In fact, Vitruvius (*ca.* 50) and Ptolemy (*ca.* 150) provided mathematical and instrumental means of calculating the sun's position for any hour, day, and observer location by the use of geometric constructions called **analemmas** (only indirectly related to the figure-8 analemma on globes). An important application of analemmas was the design of accurate horizontal and vertical direct and declining sundials for any observer location. These analemmas are awe-inspiring even today, and as the study of "Descriptive Geometry" has disappeared from our schools they can strike us as mysterious and wondrous inventions!

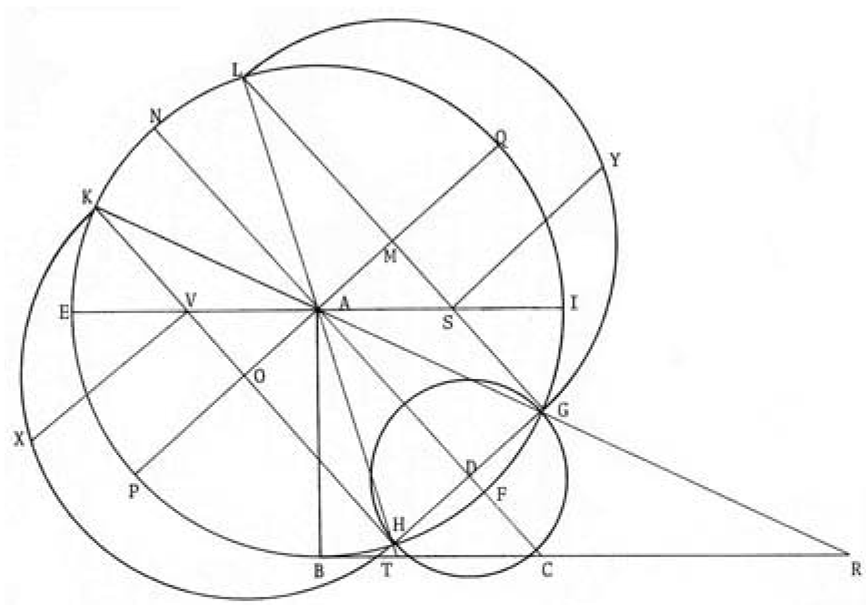


Consider the diagram at the right of the motion of the sun on the celestial sphere. NCP is the North Celestial Pole, equivalent to the North Pole of the Earth about which the Earth spins. The sun lies directly over the equinoctial line (essentially the same as the equator) on the vernal and autumnal equinoxes (~March 21 and September 21). However, since the Earth's axis is tilted by 23.5° relative to the plane of its orbit, the sun appears to move above and below the equator during the course of a year as shown by the circle in bold (the ecliptic), achieving its northerly peak at the summer solstice (~June 21) and tracing in the course of the day the circle labeled Y, and achieving its southerly peak at the winter solstice (~December 21) and tracing in the course of the day the circle labeled U. The fact that the sun is much larger than the Earth (so the rays are "parallel") justifies displacing an observer located on the top of the sphere to the middle of the sphere, simplifying the analysis. During the course of the year the observer sees the sun rise above the horizon circle from an easterly direction and trace an arc to set below the horizon circle in a westerly direction, with the summer arc higher than the winter arc. The objective is to provide a set of angles that uniquely identify the sun's position in the sky at any time; although the analemmas provided all such angles, the only two we will consider here are the zenith angle and azimuth mentioned earlier.



The shadow of the tip of a sundial will trace out an arc over the course of a day, and the path of this arc depends on the location and the day of the year. On the equinoxes, the shadow will trace a straight line (we are speaking here of latitudes between the Arctic and Antarctic Circles), while on other days the shadow will trace hyperbolas. Knowledge of the sun's position at all times provided the Greeks with the ability to mark time by drawing these shadow traces on sundials. Note that the Greek sundials marked 12 divisions of daylight, regardless of the length of daylight for that day, so-called **unequal (or seasonal) hours**.

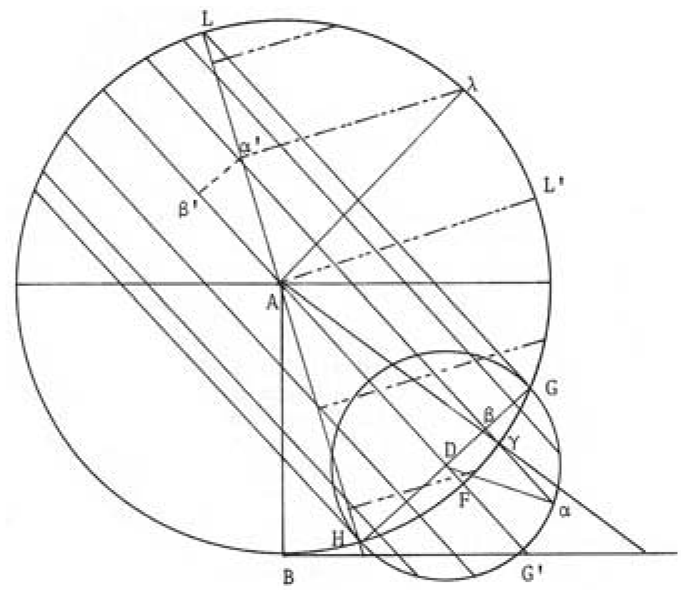
The information contained in the 3-D sphere above is neatly contained in the 2-D geometric construction below, the initial stage of the analemma described but not invented by Vitruvius. This figure is reproduced from Gibbs' book (see References). The observer is a shadow-casting stick (or **gnomon**) BA rising vertically from the ground BR on which it will cast a shadow over the course of the day. The circle about A is the meridian circle, which contains the zenith of the observer and is oriented North-South (the longitude line of the observer). BC is the shadow length of the gnomon at noon (so the sun is due south) at an equinox, so line NC represents the equator and therefore drawn at an angle equal to the latitude. The axis of the Earth PQ is constructed perpendicular to the equator line.



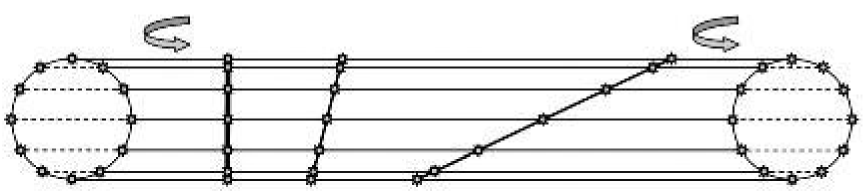
So far everything lies in the meridian plane, the plane of this sheet of paper. Now we will take the horizon circle, perpendicular to the meridian plane and moved parallel from the ground BR, and project it edge-wise into the meridian plane, resulting in EAI. Remember that this is actually a circle extending through this paper, as this will become important.

Vitruvius marked off 1/15 of the meridian circle (or 24° rather than the more correct 23.5° ecliptic angle used by Ptolemy) on either side of the intersection of the equator with the meridian circle at F (at H and G) and drew the small corresponding circle centered at F. Then because H and G are located at the extreme angles of the ecliptic, KH (drawn parallel to the equator) is the projection into the meridian plane of the circle the sun takes in at the winter solstice and LG is the same for the summer solstice. We can now mark the points of two shadows of the gnomon *due south at local noon*: T in the summer from extending LAH, and R in the winter from extending KAG. LAH and KAG represent the ecliptic. Also, remembering that LG, KH and all other projections of the parallel circles of the sun's path between these extremes, we can take, say, the summer sun circle and swing it about LG into the meridian plane, becoming the half-circle LYG. Y is found by constructing a perpendicular to LG from the point S where LG intersects the horizon EAI. Now since EAI and LG are projections of circles that are actually at right angles to this sheet of paper, do you see that when LYG is swung back out of this paper to its proper position the point S represents the point where the sun meets the horizon, i.e., sunset? Therefore, LY is the half-daylight duration and the fraction LY/LG represents the fraction of day to night for the summer solstice. The figure also shows the same construction for the winter solstice.

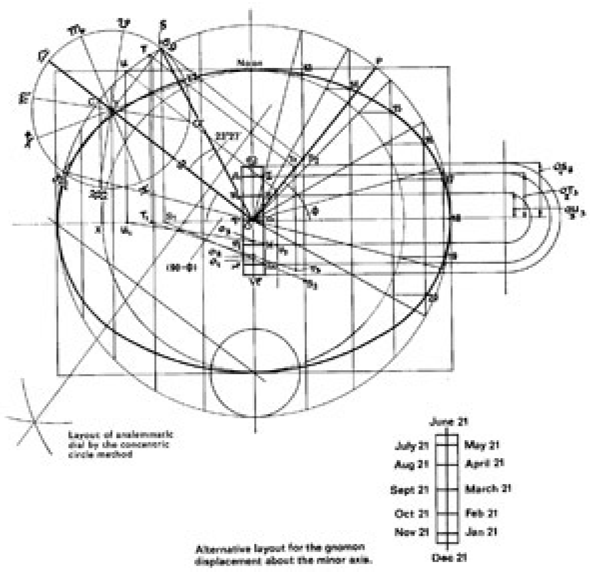
We can find the sun circles for other days between the two solstices. Again considering the same summer sun circle on LG swung around into the meridian plane, as shown in the figure on the right, we can mark off 6 equal arcs along LG to represent the 6 signs of the zodiac that the sun passes through in a half-year, and we can drop perpendiculars to the diameter LH to find the sun circles for the days of entries into those signs (and others days). The other 6 signs produce the same lines as the first 6 lines. Constructing lines parallel to the equator at these ecliptic longitudes provide noon shadow tips on the ground BR for those days.



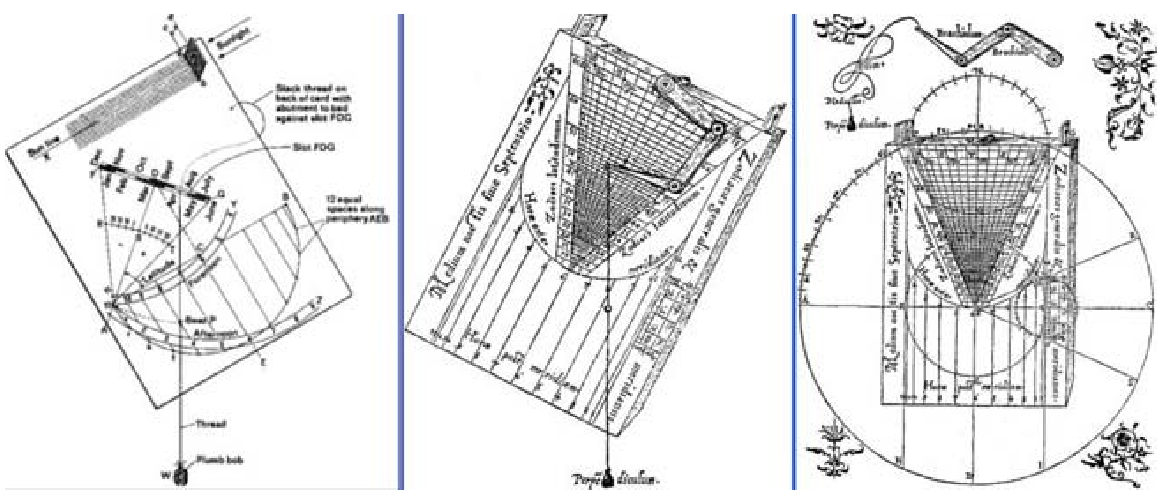
However, it turns out that dividing the half-circumference of the small circle into 6 parts and creating parallels to the equator produces the same sun circles, and this is such a useful trick that this small circle has its own name, the **menaeus**. Gibbs, Heilbron and Drecker provide proofs of this, but in fact it is easy to see. Referencing my figure below, horizontal lines are drawn from 12 equidistant locations on the menaeus on the left. The menaeus rotated edge-on is shown, then two edge-on and one facing ecliptic circles oriented at different angles relative to the menaeus but having the same diameter projection.



As you can see, angling the ecliptic spreads the sun positions proportionally, so in fact a menaeus can be drawn at any angle to the ecliptic as long as its diameter equals the projected length of the ecliptic. Lines parallel to the equator have to be drawn at the points on the ecliptic, though, so having the menaeus centered on the equator provides these parallel lines directly. Also, this location does not significantly interfere with the other lines of the analemma, allowing the menaeus to consist of a separate attachment to an instrument, permanently subdivided.

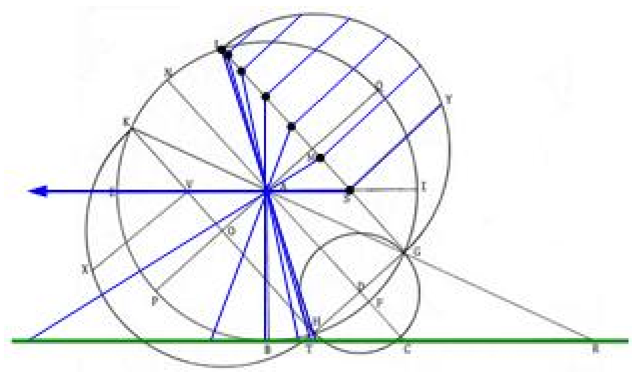


The menaeus is a trick that you may notice now and again in some geometric constructions. The menaeus also makes an appearance in the figure on the left demonstrating the construction of the elliptical trace on an analemmatic sundial. This type of sundial has a vertical gnomon but provides the time in modern equal hours, requiring the gnomon to be moved to a specific offset for a given day. Its construction is based on the analemma of Vitruvius, hence its name. The menaeus actually makes an appearance and functions in at least two other portable sundials that are based on the sun's altitude and the day of the year. The figure below shows the Capuchin card dial (located on the left), as well as the Regiomontanus dial and its very interesting construction (from Drinkwater).

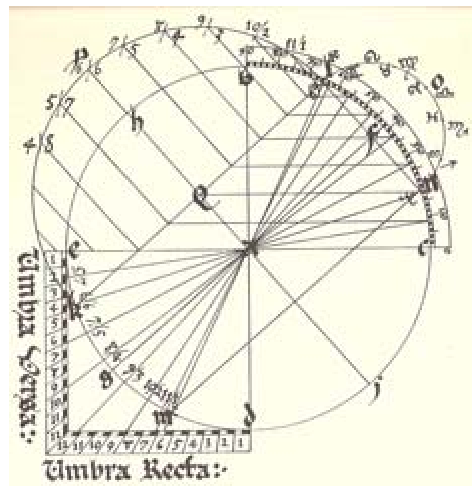


So how do we find the sun's position for times other than noon on a given day? We had seen earlier that the line from the upper end of the sun circle for the day through the tip of the gnomon onto the horizon provides the noon shadow length on that day (due South) measured from the base of the gnomon. In fact, we can find the South component of the shadow length for *any* hour of the day by dropping a line from the hour point on the sun circle perpendicular to its axis and projecting this axis point through the gnomon tip to the baseline, as shown in the next figure.

In this figure the arc from noon to sunset is divided into 6 equal lengths (giving seasonal hours), and the black dots represent the perpendicular projections onto the axis of the sun circle for the day (here the summer solstice). Blue lines are drawn from these points to the green baseline through the top of the gnomon, and the distance from the base of the gnomon to these intersections provides for every hour the South component of the shadow length for the given height of the gnomon. At sunset the sun is on the horizon, so that blue line is horizontal.



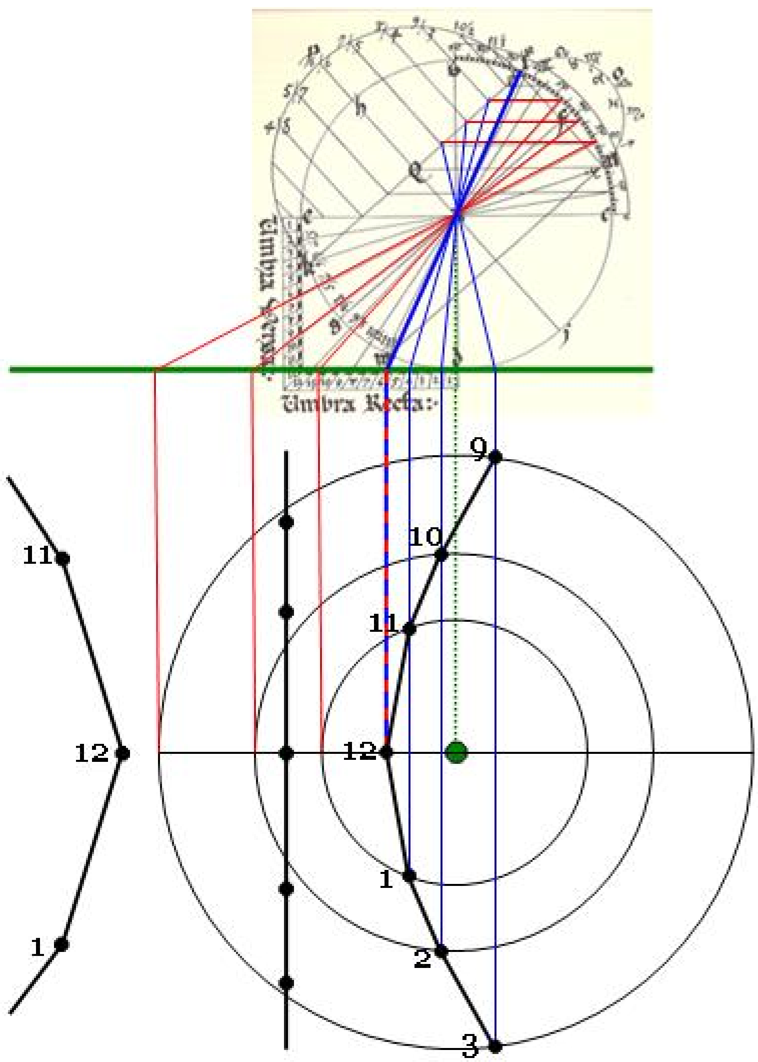
But we need to know more than just the South component of the shadow length from to plot the day curves. Look at the figure below with the ivory background (also displayed in my "Welcome" post), reproduced from Peter Drinkwater's translation and interpretation of the 16th century works of Oronce Fine. The figure is flipped left-right from that above, and the menaeus is at the upper end, but you should recognize some things here (ignore the radiating lines from the center for the moment). See the menaeus, the projections of the sun circles for the equinoxes and solstices, and the winter solstice half-circle swung upward into the meridian plane?



Here the quarter-circle up from the horizon (*eac* here) is divided into 90°. The sun half-circle at the summer solstice that has been swung into the meridian plane is divided into 12 parts (one per hour), which makes sense because the axis of the circle is normally parallel to with the Earth's axis about which the sun seems to rotate. At the points on the axis where the perpendiculars to the hour lines meet, lines parallel to the horizon are drawn to the degree scale, giving the sun's altitude at that hour on that day. If you remember the sun's half-circle as actually perpendicular out of this sheet of paper, you can visualize why this would be so—drawing the parallel lines is equivalent to rotating the meridian circle about the center to become a circle through East, West and the zenith points. Note that **equal hours** are measured here, while the Greeks and Romans would simply divide the daylight hours into 12 parts—since we are seeing only a half circle here, the arc between the "4/8" mark (equivalent to sunset point S in the earlier diagram) and the almost hidden "12" mark at the upper termination would be divided into 6 parts to mark the unequal hours as we did earlier.

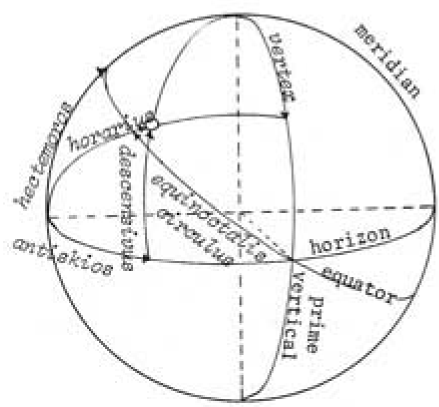
You can see that drawing lines from these altitude degree readings through the center point provides shadow lengths along the ground at the base of the vertical gnomon *da* for every hour of the day. In this case, however, the author chose to provide readings on a **shadow square**, a two-dimensional scale that provides the equivalent of the tangent of the angle as the ratio of the height and vertical distances—for example, typically there are 12 units displayed in each dimension to provide simple fractions, so the tangent of the ray that is shown intersecting the shadow square 8 units from the bottom right is 12/8 and the tangent of the ray intersecting 6 units from the top is 6/12. The shadow square is generally seen under the alidade (the sight) on the back of astrolabes, which is useful because the tangent of a sighted angle to the top of a building, say, can be used to determine the height of the building by multiplying this tangent by the measured distance to the building.

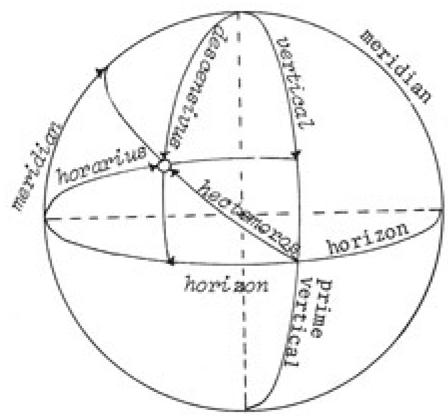
But as I mentioned, extending these lines through the shadow square onto the baseline provides the absolute length of the gnomon shadow, measured from its base, for the day corresponding to the sun circle. So we have the situation where we know the South component of the gnomon shadow for any hour of the day (by drawing lines from the hour point on the sun circle perpendicular to its axis, then from this point through the gnomon tip to the baseline, shown here in blue), and we know the full shadow length (by extending the points on the altitude quarter-circle through the gnomon tip, shown here in red). We can construct the sundial layout, viewed from above, directly below the analemma. We first mark a point for the gnomon (here in green) below that of the analemma. Then for each hour we drop a line corresponding to the South component of the shadow (a blue line), and we drop a line corresponding to the full shadow length (a red line). We draw circles centered on the gnomon with radii equal to the full shadow lengths, and the intersections of these circles with the South component lines provide the locations of the tip of the gnomon shadow for every hour of that day. On the equinoxes these points form a straight East-West line a distance from the



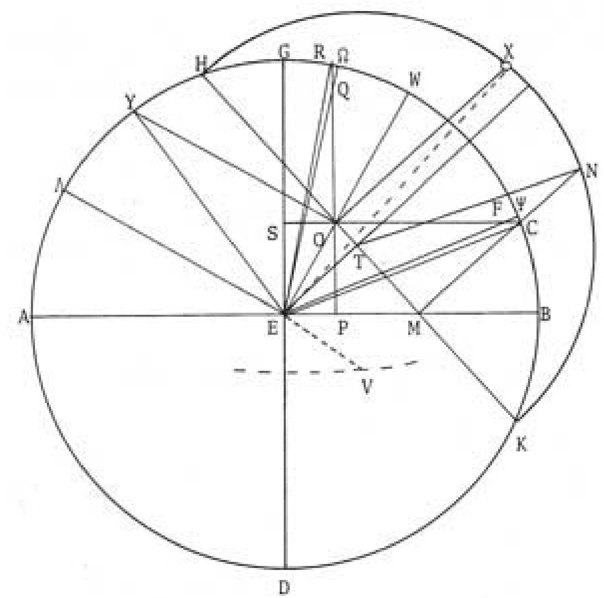
gnomon base equal to the extension of the equator line to the baseline; otherwise a hyperbola results. Drawing lines through shadow points for the same hour on different days (i.e., from different sun circles) provides the hour lines of the sundial—they are slightly curved, although all surviving Greek planar dials simply have straight lines between the points for the two solstices. Evans provides examples of constructing these layouts, including tips such as drawing the baseline higher than the base of the meridian circle in order to work with a larger meridian circle without having such long projected lengths on the baseline.

A century later comes Ptolemy, who finds the spherical model used by "the ancients" to be inelegant, as the equator in the first figure to the right has a coordinate system that depends on the geographic location of the observer. (By the way, all these angles shown in this figure were derivable by those ancients!). Ptolemy replaced the equatorial circle with a moveable circle that passes through the East and West points and the sun (the *hectemoros circle*), as shown in the rightmost figure, and in the process created a new analemma.





Ptolemy's analemma (from his text *Analemma*) is shown below, again from Gibbs. Similar components to Vitruvius' analemma include the meridian circle centered on E, the horizon AB, the half-circle HXK of the sun's path swung into the meridian plane, and the daylight fraction HN/NK.

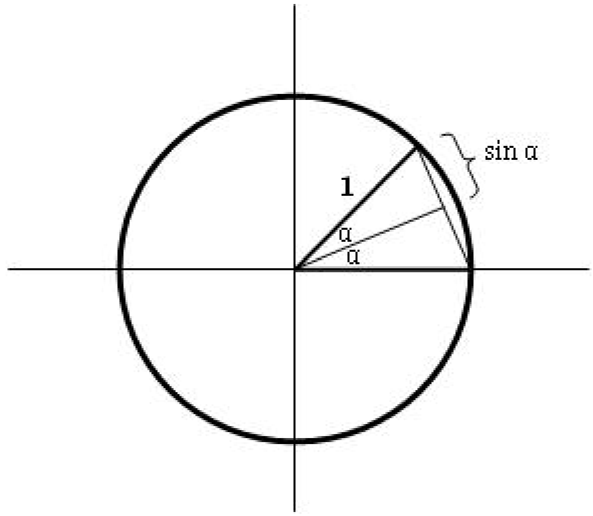


The rest is a compression of Gibbs' text:

> The sun X is at the second hour point here, and O is found on HK by drawing a perpendicular from X. Y on the meridian circle is found such that OY equals XO, and ΛE is drawn perpendicular to EO. POR and SOC are drawn perpendicular to EB and EG, respectively. Q is marked on POR such that PQ=OX, and point F is marked on SOC such that SF=OX. Then the hectemoros angle is equal to ΛEY; the horarius angle is equal to GEC; the meridian angle is equal to BEW; the vertical angle is equal to GEΨ; the horizon angle is equal to GEΩ.

Drecker and Gibbs provide the derivations, and in fact all the angles are derivable geometrically from this analemma, such as the zenith distance and azimuth angle whose formulas from spherical trigonometry are given at the top of this post. Ptolemy then created a mechanism to rapidly create these analemmas for different latitudes and solar declinations. Elements that are unchanged, such as the meridian circle and parallels to the equator, were engraved or painted on metal, stone or wood along with scales for major latitudes and equinox hour lines. A wax layer was applied to the plate to allow date- and location-specific lines to be drawn, and by rotating the disk and using a right-angled straightedge, the solutions could be quickly found. Neugebauer conjectures that the Greek interest in conic sections originated in the study of sundials, and that their interest in trisecting an angle with compass and straightedge was related to dividing the analemma circles into 12 parts, i.e., two bisections and then a trisection within each quarter-circle. I assume he means dividing the daylight portions of the sun circles into 12 parts, as dividing a full circle (such as the menaeus) into 12 parts is easy because marking off a chord length equal to half the radius will span 60° of the circle.

Actually, the way that the modern trigonometric calculations were embedded in the geometric constructions is through the chords or half-chords that are drawn between points on the circles. As shown in the figure on the right, sin α = (chord 2α) / 2. In the sexagesimal number system of Ptolemy's day, sin α = (chord 2α) / 120. From this figure it is also apparent that cos α = [chord (180° - 2α)] / 2. As an aside, to create a menaeus with a span +/-23.5°, I would propose marking off on the meridian circle 2/5 of the meridian radius, as it turns out that the chord formula for this span yields a menaeus diameter that is 0.3987 of the meridian radius.



As an example of solving trigonometric formulas using chords, it is possible to find the latitude φ of a location by the length of the longest day (at the summer solstice) through the formula tan φ = (-cos (M/2) / tan ε), where M is the length of the longest day converted to an angle at 15° per hour and ε is the obliquity of the ecliptic (23.5°). The day lengths would have been measured at the time with water clocks (**clepsydras**) that drained water to measure time. In the first Vitruvius analemma figure above, angle NAL=ε and angle MAS=φ, with the sun circle LYG representing the summer solstice. The entire figure can be drawn other than the horizon lines EAI and BR that specify the latitude, setting Y such that the length LY/LG represents the half-daylight fraction observed for that day. Then the horizon line at A is drawn along AS, completing the analemma and identifying the latitude without solving the trigonometric formula directly (see Wilson). Rather than an absolute length of the longest day, it is also possible to perform this construction with the ratio of the lengths of the longest day to the shortest day. This is more than a passing curiosity—the Greeks used the ratios of longest to shortest days to map the world. In fact, Neugebauer tells us that the ancient Babylonians accepted the round value 3:2 as this ratio, apparently without realizing the geographic variation, and this led the Greeks to calculate the **clima** (a zone roughly equivalent to latitude) of Babylon at what would be 35° rather than the correct 32.5°, "seriously affecting the shape of the eastern part of the ancient map of the world."

I find all this intriguing from a creative as well as a technical aspect. To my knowledge, the only prevalent use of geometric constructions today is in graphical layouts of sundials. Cousins' book on sundials is chock-full of these fascinating constructions, and the best part is that when you are finished with the design and have built the physical sundial, you can take your layout sheet, frame it, and mount it on the wall as a piece of art.

References

Calvert, J.B. *Vitruvius and the Analemma*, at http://mysite.du.edu/~etuttle/classics/analemma.htm. Discusses the analemma from the point of view of orthographic projection.

Cousins, Frank W. Sundials: **The Art and Science of Gnomonics**. New York: Pica Press, 1970. This book on sundials has a particular focus on using geometric constructions to understand and lay out all manner of dials.

Drecker, Joseph. **Theorie der Sonnenuhren**. Berlin, 1925. I have not seen this work, but Gibbs and Neugebauer both refer to this work for more derivations related to the analemma of Ptolemy.

Drinkwater, Peter I. **Oronce Fine's Second Book of Solar Horology**. Warwickshire, England: Self-published, 1993. This is a unique booklet, a translation and interpretation of this work of Oronce Fine (1494-1555), Dialist to Francis I of France. The figure with the ivory background is reproduced from this work, although it is not clear how much of this figure is due to Fine and how much to Drinkwater.

Evans, James. **The History and Practice of Ancient Astronomy**. New York and Oxford: Oxford University Press, 1998. This is a fantastic book that I recommend to anyone with an interest in this topic (it can be found on Amazon.com at http://www.amazon.com/gp/product/0195095391/ref=s9_asin_image_1/102-7664496-4636903?pf_rd_m=ATVPDKIKX0DER&pf_rd_s=center-2&pf_rd_r=013CV68FVD9Q8XW56A6F&pf_rd_t=101&pf_rd_p=278240301&pf_rd_i=507846). It also includes, for example, templates and instructions for creating your own astrolabe and an equatorium for the position of Mars. Perhaps it's a good thing I found this reference so late in preparing this essay—I learned more about the analemmas by working hard to figure them out, and I would not have located most of the other interesting references here.

Gibbs, Sharon L. **Greek and Roman Sundials.** New Haven and London: Yale University Press, 1976. This was my primary source for detailed information on the analemmas of Vitruvius and Ptolemy, as well as the source of a few of the diagrams.

Heilbron, J.L. **The Sun in the Church: Cathedrals as Solar Observatories.** Cambridge and London: Harvard University Press, 1999. A very interesting book that briefly describes the analemma of Vitruvius and gives a short proof of why the menaeus works in Appendix B. The figure of the sun's motion on the celestial sphere is reproduced from this book. See also *A Mathematical Supplement to "The Sun in the Church: Cathedrals as Solar Observatories"* by Ng Yoke Leng and Helmer Aslaksen at http://www.math.nus.edu.sg/aslaksen/projects/heilbron/.

Neugebauer, O. **The Exact Sciences in Antiquity, 2nd Ed.** New York: Dover Publications, 1969. This book has a wider scope, but a brief discussion of Ptolemy's analemma and its mechanization into an instrument appears in Appendix II, *On Greek Mathematics*. A more comprehensive journal article on the topics of this appendix can be found online at http://www.ams.org/bull/1948-54-00/S0002-9904-1948-09089-9/S0002-9904-1948-09089-9.pdf.

Neugebauer, O. **Astronomy and History: Selected Essays.** New York: Springer-Verlag, 1983. This book contains is a collection of papers by the author over many years. Some information overlaps that of the previous reference (and in fact it contains the journal article linked in the previous reference), but this book is my source on the mistaken calculation of Babylon's location (pp. 64,240).

Wilson, Curtis. *Hipparchus and Spherical Trigonometry*, article #2 of http://www.dioi.org/vols/w71.pdf. Describes how the analemma incorporates the formula for the latitude based on the length of the longest day. (I'm not knowledgeable enough to recommend other articles of this journal, but they do make sensational reading as all-out academic battles.)