Can the equation \( z = xy + \sqrt{1+x^2}\sqrt{1+y^2} \) be constructed as a parallel-scale nomogram? We will use Saint-Robert’s criterion.

Here \( F(x,y,z) = -z + xy + \sqrt{1+x^2}\sqrt{1+y^2} = 0 \).

\[
\frac{\partial F}{\partial x} = y + \sqrt{1+y^2}(1/2)(1+x^{-2})^{-1/2}(2x) = y + x\sqrt{1+y^2}\frac{1}{1+x^2}
\]

\[
\frac{\partial F}{\partial y} = x + \sqrt{1+x^2}(1/2)(1+y^{-2})^{-1/2}(2y) = x + y\sqrt{1+x^2}\frac{1}{1+y^2}
\]

\[
R = \frac{\partial F}{\partial x} = \frac{y + x\sqrt{1+y^2}}{x + y\sqrt{1+x^2}} = \sqrt{\frac{1+y^2}{1+x^2}}
\]

\[
\ln R = \frac{1}{2}\ln(1+y^2) - \frac{1}{2}\ln(1+x^2)
\]

\[
\frac{\partial \ln R}{\partial x} = \frac{1}{2} \left( \frac{1}{1+x^2} \right)^2 (2x) = -\frac{x}{1+x^2}
\]

\[
\frac{\partial^2 \ln R}{\partial x \partial y} = 0
\]

This result means we can represent \( F(x,y,z) = -z + xy + \sqrt{1+x^2}\sqrt{1+y^2} = 0 \) as a parallel-scale nomogram, or in other words, we can express it in the form \( Z(z) = X(x) + Y(y) \) or in the form \( Z(z) = X(x)Y(y) \) which can be rewritten as \( \ln Z(z) = \ln X(x) + \ln Y(y) \).

To find \( X(x) \),

\[
\ln \frac{dX}{dx} = \int \frac{\partial \ln R}{\partial x} dx = \int \frac{x}{1+x^2} dx = -\frac{1}{2} \ln(1+x^2) = \ln \left( \frac{1}{\sqrt{1+x^2}} \right)
\]

so

\[
\frac{dX}{dx} = \frac{1}{\sqrt{1+x^2}}
\]

from integral tables, \( X = \ln(x + \sqrt{1+x^2}) \) which is sometimes given as \( \sinh^{-1} x \)
For $Y(y)$,
\[
\frac{dY}{dy} = \frac{dX/dx}{R} \quad \text{which will contain no variable } x \\
= \frac{1}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+y^2}}
\]
so $Y = \ln(y + \sqrt{1+y^2})$

And for $Z(z)$, we can use $Z(z) = X(x) + Y(y)$:
\[
Z(z) = \ln(x + \sqrt{1+x^2}) + \ln(y + \sqrt{1+y^2}) \\
= \ln \left[(x + \sqrt{1+x^2})(y + \sqrt{1+y^2})\right]
\]

We are guaranteed that we can use $z = xy + \sqrt{1+x^2}\sqrt{1+y^2}$ to express $Z(z)$ in terms of $z$ and eliminate all $x$ and $y$ terms. With some algebra we can find that
\[
Z(z) = \ln(z + \sqrt{z^2-1})
\]

Substituting $X$, $Y$ and $Z$ into $Z(z) = X(x) + Y(y)$ we have
\[
\ln(z + \sqrt{z^2-1}) = \ln(x + \sqrt{1+x^2}) + \ln(y + \sqrt{1+y^2}) \\
\text{or} \quad (z + \sqrt{z^2-1}) = (x + \sqrt{1+x^2})(y + \sqrt{1+y^2})
\]
which is the form for a nomogram consisting of three parallel scales.

It turns out that this works when $x$ in the original equation $F(x,y,z)$ is replaced with any function of $x$, $y$ is replaced with any function of $y$, and $z$ is replaced with any function of $z$. It is a surprising result, and one that Maurice d’Ocagne included in his 1899 book, *Traité de Nomographie* (pages 418-421).